EXAMPLE 1 Graph a rational function of the form $y = \frac{a}{x}$

Graph the function $y = \frac{6}{x}$. **Compare the graph with the graph of** $y = \frac{1}{x}$.

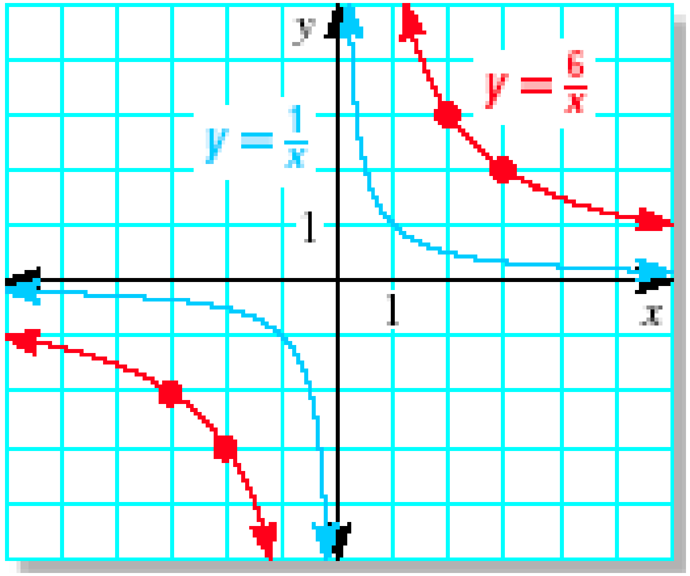
SOLUTION

STEP 1

Draw the asymptotes $x = 0$ **and** $y = 0$.

STEP 2

Plot points to the left and to the right of the vertical asymptote, such as $(-3, -2)$, $(-2, -3)$, $(2, 3)$, and $(3, 2)$.



EXAMPLE 1 Graph a rational function of the form $y = \frac{a}{x}$

STEP 3

Draw the branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

The graph of $y = \frac{6}{x}$ lies farther from the axes than the graph of $y = \frac{1}{x}$.

Both graphs lie in the first and third quadrants and have the same asymptotes, domain, and range.

EXAMPLE 2 Graph a rational function of the form $y = \frac{a}{x-h} + k$

Graph $y = \frac{-4}{x+2} - 1$. State the domain and range.

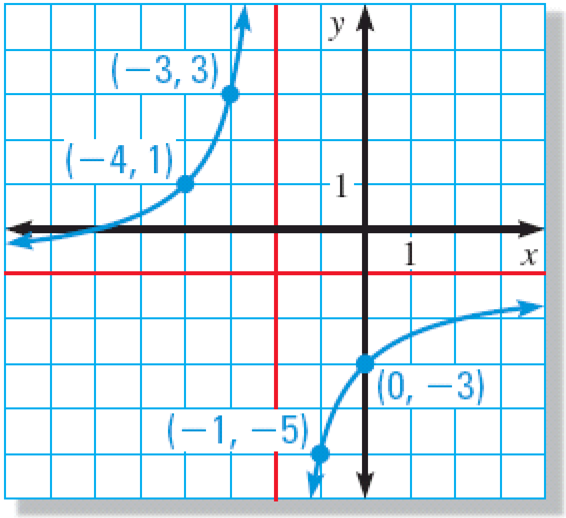
SOLUTION

STEP 1

Draw the asymptotes $x = -2$ and $y = -1$.

STEP 2

Plot points to the left of the vertical asymptote, such as $(-3, 3)$ and $(-4, 1)$, and points to the right, such as $(-1, -5)$ and $(0, -3)$.



EXAMPLE 2

Graph a rational function of the form $y = \frac{a}{x-h} + k$

STEP 3

Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

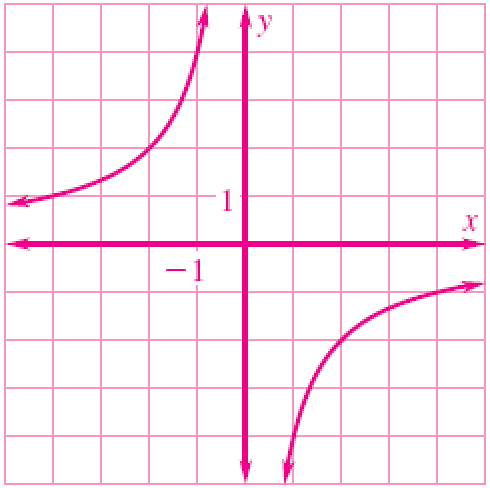
The domain is all real numbers except -2, and the range is all real numbers except -1.

GUIDED PRACTICE for Examples 1 and 2

Graph the function. State the domain and range.

1. $f(x) = \dfrac{-4}{x}$

SOLUTION



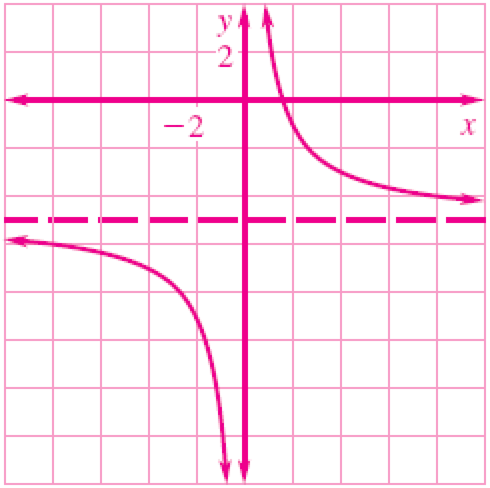
ANSWER **domain: all real numbers except** 0**,**
range: all real numbers except 0**.**

GUIDED PRACTICE for Examples 1 and 2

Graph the function. State the domain and range.

2. $y = \frac{8}{x} - 5$

SOLUTION



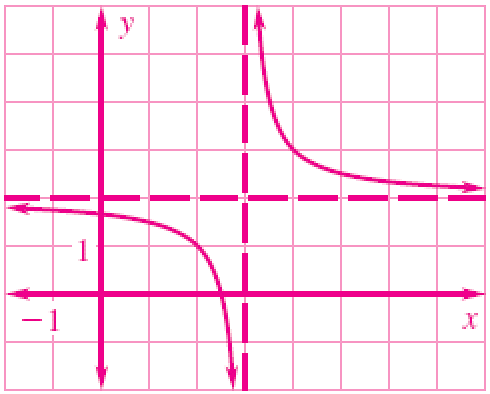
ANSWER **domain: all real numbers except** 0**, range: all real numbers except** -5**.**

GUIDED PRACTICE for Examples 1 and 2

Graph the function. State the domain and range.

3. $y = \dfrac{1}{x-3} + 2$

SOLUTION



ANSWER **domain: all real numbers except 3, range: all real numbers except 2.**

EXAMPLE 3 Graph a rational function of the form $y = \frac{ax + b}{cx + d}$

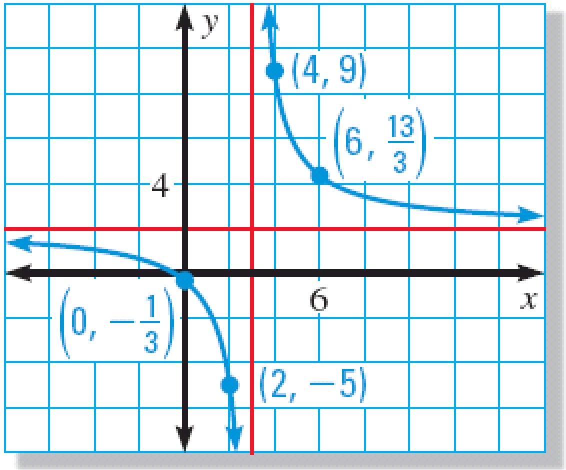
Graph $y = \frac{2x + 1}{x - 3}$ **. State the domain and range.**

SOLUTION

STEP 1

Draw the asymptotes. Solve $x - 3 = 0$ **for** x **to find the vertical asymptote** $x = 3$**.The horizontal asymptote is the line**

$$y = \frac{a}{c} = \frac{2}{1} = 2$$



EXAMPLE 3 Graph a rational function of the form $y = \frac{ax + b}{cx + d}$

STEP 2

Plot points to the left of the vertical asymptote, such as $(2, -5)$ and $\left(0, -\frac{1}{3}\right)$, and points to the right, such as $(4, 9)$ and $\left(6, \frac{13}{3}\right)$.

STEP 3

Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

ANSWER

The domain is all real numbers except 3.
The range is all real numbers except 2.

EXAMPLE 4 Solve a multi-step problem

3-D Modeling

A 3-D **printer builds up layers of material to make three dimensional models. Each deposited layer bonds to the layer below it. A company decides to make small display models of engine components using a** 3-D **printer. The printer costs** $24,000. **The material for each model costs** $300.

- **Write an equation that gives the average cost per model as a function of the number of models printed.**

EXAMPLE 4 Solve a multi-step problem

- **Graph the function. Use the graph to estimate how many models must be printed for the average cost per model to fall to $700.**

- **What happens to the average cost as more models are printed?**

SOLUTION

STEP 1

Write a function. Let c be the average cost and m be the number of models printed.

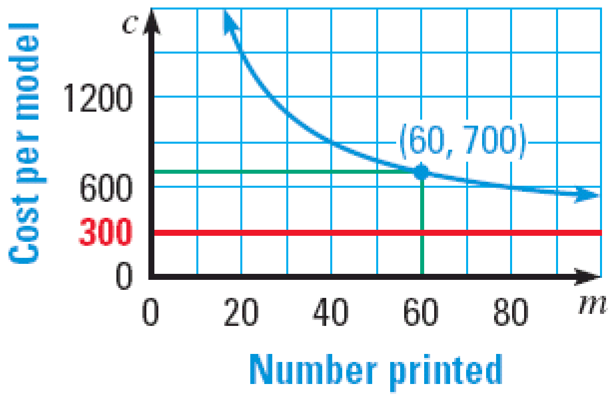
$$c = \frac{\text{Unit cost} \cdot \text{Number printed} + \text{Cost of printer}}{\text{Number printed}}$$

$$= \frac{300m + 24{,}000}{m}$$

EXAMPLE 4 Solve a multi-step problem

STEP 2

Graph the function. The asymptotes are the lines $m = 0$ and $c = 300$. The average cost falls to $\$700$ per model after 60 models are printed.



STEP 3

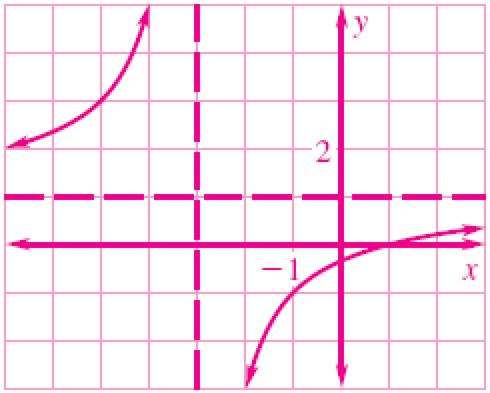
Interpret the graph. As more models are printed, the average cost per model approaches $\$300$.

GUIDED PRACTICE for Examples 3 and 4

Graph the function. State the domain and range.

4. $y = \dfrac{x-1}{x+3}$

SOLUTION



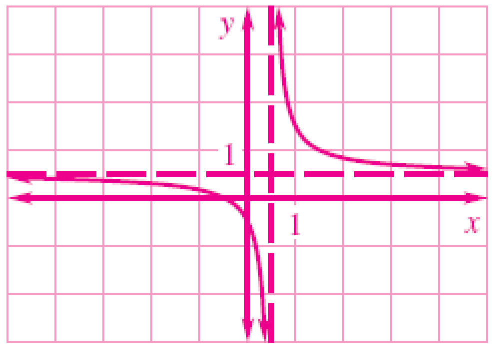
ANSWER **domain: all real numbers except** -3**,**
range: all real numbers except 2**.**

GUIDED PRACTICE for Examples 3 and 4

Graph the function. State the domain and range.

5. $y = \dfrac{2x+1}{4x-3}$

SOLUTION



ANSWER **domain: all real numbers except** $\frac{1}{2}$**,**

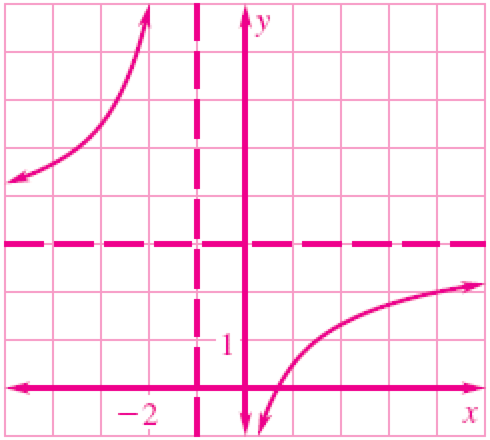
range: all real numbers except $\frac{1}{2}$**.**

GUIDED PRACTICE for Examples 3 and 4

Graph the function. State the domain and range.

6. $f(x) = \dfrac{-3x+2}{-x-1}$

SOLUTION



ANSWER **domain: all real numbers except** -1 **,**
range: all real numbers except 3 **.**

GUIDED PRACTICE for Examples 3 and 4

7. **What If? In Example 4, how do the function and graph change if the cost of the 3-D printer is** $21,000?

ANSWER

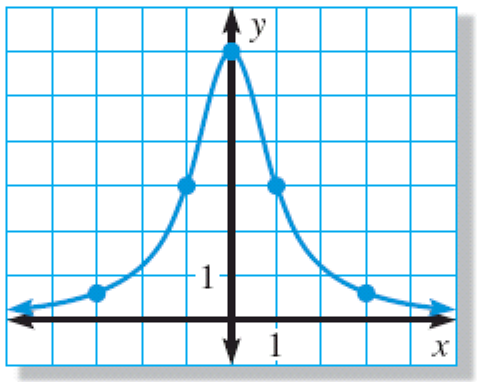
Sample answer: **In the function, 24,000 is replaced by 21,000. On the graph, the asymptotes remain at $m = 0$ and $c = 300$, but the values decrease from a smaller starting point.**

EXAMPLE 1

Graph a rational function ($m < n$)

Graph $y = \dfrac{6}{x^2 + 1}$. State the domain and range.

SOLUTION



The numerator has no zeros, so there is no x-intercept. The denominator has no real zeros, so there is no vertical asymptote.

The degree of the numerator, 0, is less than the degree of the denominator, 2. So, the line $y = 0$ (the x-axis) is a horizontal asymptote.

EXAMPLE 1 Graph a rational function ($m < n$)

The graph passes through the points (–3, 0.6), (–1, 3), (0, 6), (1, 3), **and** (3, 0.6). **The domain is all real numbers, and the range is** $0 < y \leq 6$.

EXAMPLE 2 Graph a rational function ($m = n$)

Graph $y = \dfrac{2x^2}{x^2 - 9}$.

SOLUTION

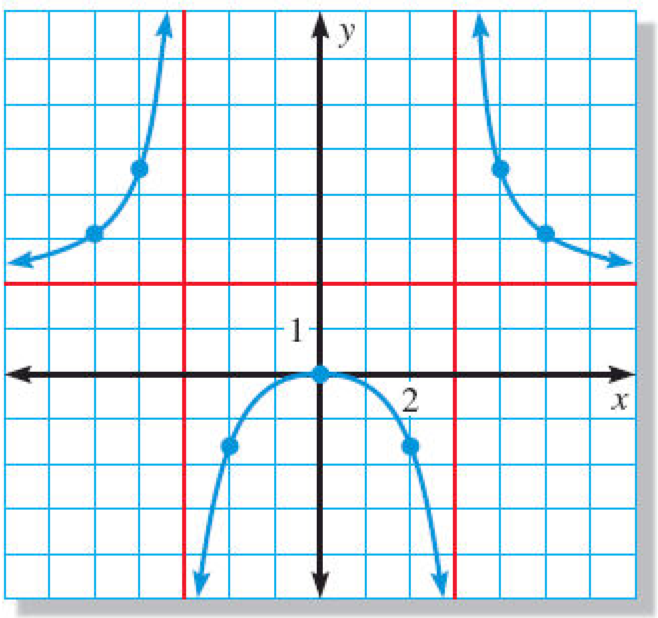
The zero of the numerator $2x^2$ is 0, so 0 is an x-intercept. The zeros of the denominator $x^2 - 9$ are ± 3, so $x = 3$ and $x = -3$ are vertical asymptotes.

The numerator and denominator have the same degree, so the horizontal asymptote is $y = \dfrac{a_m}{b_n} = \dfrac{2}{1} = 2$

Plot points between and beyond the vertical asymptotes.

EXAMPLE 2 Graph a rational function ($m = n$)

	x	y
To the left of $x = -3$	−5	3.1
	−4	4.6
Between $x = -3$ and $x = 3$	−2	−1.6
	0	0
	2	−1.6
To the right of $x = 3$	4	4.6
	5	3.1



EXAMPLE 3 Graph a rational function ($m > n$)

Graph $y = \dfrac{x^2 + 3x - 4}{x - 2}$.

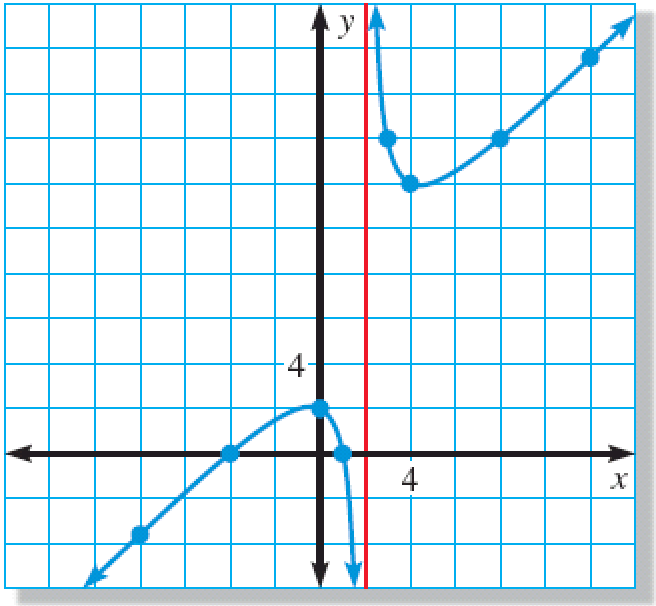
SOLUTION

The numerator factors as $(x + 4)(x - 1)$, so the x-intercepts are -4 and 1. The zero of the denominator $x - 2$ is 2, so $x = 2$ is a vertical asymptote.

The degree of the numerator, 2, is greater than the degree of the denominator, 1,so the graph has no horizontal asymptote. The graph has the same end behavior as the graph of $y = x^{2-1} = x$. Plot points on each side of the vertical asymptote.

EXAMPLE 3 Graph a rational function ($m > n$)

	x	y
To the left of $x = 2$	−8	−3.6
	−4	0
	0	2
	1	0
To the right of $x = 2$	3	14
	4	12
	8	14
	12	17.6

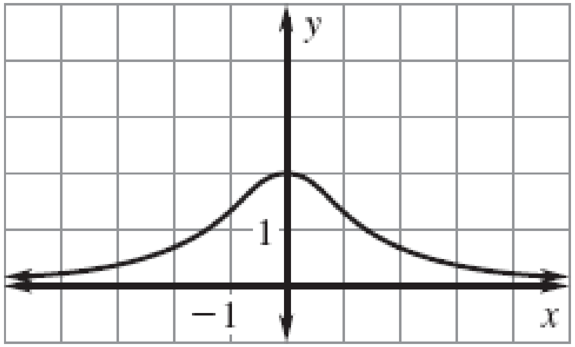


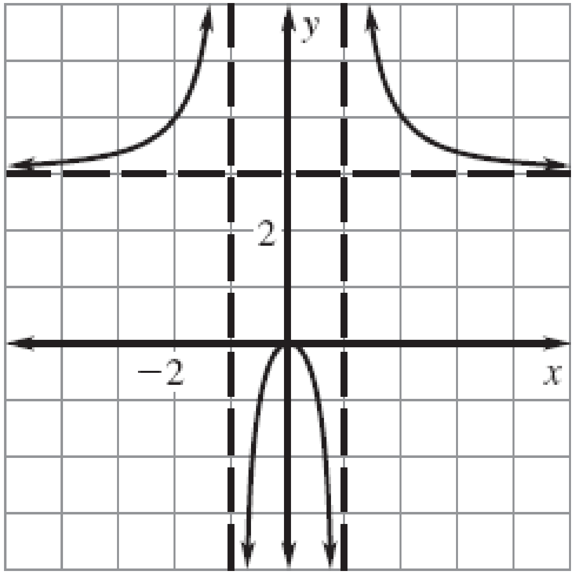
GUIDED PRACTICE for Example 1, 2 and 3

Graph the function.

1. $y = \dfrac{4}{x^2 + 2}$

2. $y = \dfrac{3x^2}{x^2 - 1}$





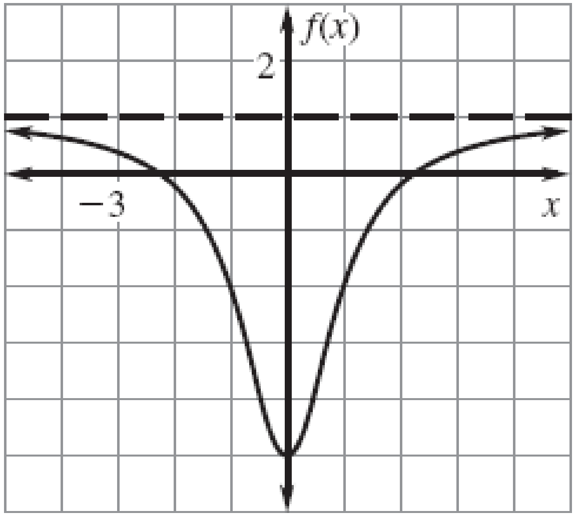
http://www.classzone.com/cz/books/algebra_2_2011_na/book_home.htm

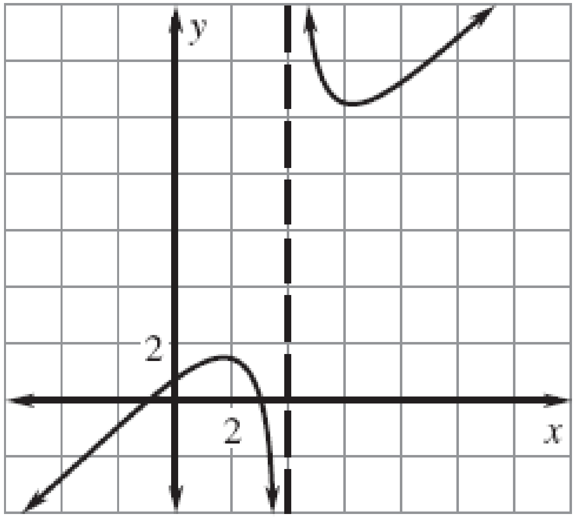
GUIDED PRACTICE for Example 1, 2 and 3

Graph the function.

3. $f(x) = \dfrac{x^2 - 5}{x^2 + 1}$

4. $y = \dfrac{x^2 - 2x - 3}{x - 4}$





EXAMPLE 4

Solve a multi-step problem

Manufacturing

A food manufacturer wants to find the most efficient packaging for a can of soup with a volume of 342 cubic centimeters. Find the dimensions of the can that has this volume and uses the least amount of material possible.

EXAMPLE 4 Solve a multi-step problem

SOLUTION

STEP 1

Write an equation that gives the height h of the soup can in terms of its radius r. Use the formula for the volume of a cylinder and the fact that the soup can's volume is 342 cubic centimeters.

$V = \pi r^2 h$ — Formula for volume of cylinder

$342 = \pi r^2 h$ — Substitute 342 for V.

$\frac{342}{\pi r^2} = h$ — Solve for h.

EXAMPLE 4 Solve a multi-step problem

STEP 2

Write a function that gives the surface area S of the soup can in terms of only its radius r.

$S = 2\pi r^2 + 2\pi rh$ **Formula for surface area of cylinder**

$= 2\pi r^2 + 2\pi r\left(\frac{342}{\pi r^2}\right)$ **Substitute $\frac{342}{\pi r^2}$ for V.**

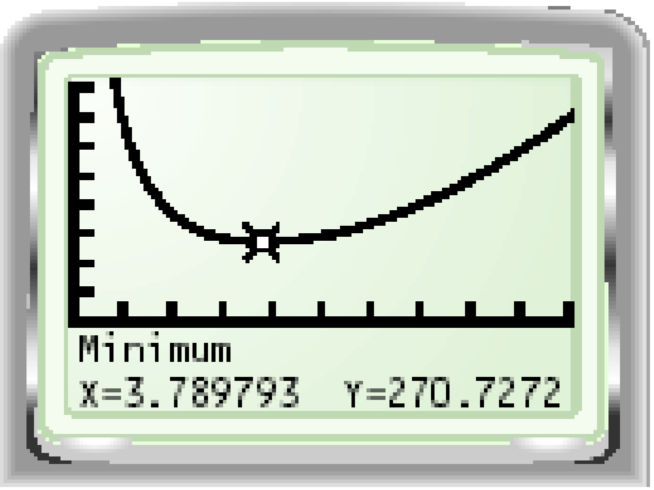
$= 2\pi r^2 + \frac{684}{r}$ **Solve for h.**

EXAMPLE 4 Solve a multi-step problem

STEP 3

Graph the function for the surface area S using a graphing calculator. Then use the *minimum* feature to find the minimum value of S.

You get a minimum value of about 271, which occurs when $r \approx 3.79$ and



$$h \approx \frac{342}{\pi(3.79)^2} \approx 7.58.$$

EXAMPLE 4 Solve a multi-step problem

ANSWER

So, the soup can using the least amount of material has a radius of about 3.79 centimeters and a height of about 7.58 centimeters. Notice that the height and the diameter are equal for this can.

GUIDED PRACTICE for Example 4

5. **What If? In Example 4, suppose the manufacturer wants to find the most efficient packaging for a soup can with a volume of 544 cubic centimeters. Find the dimensions of this can.**

ANSWER

$r \approx 4.42$cm

$h \approx 8.86$cm