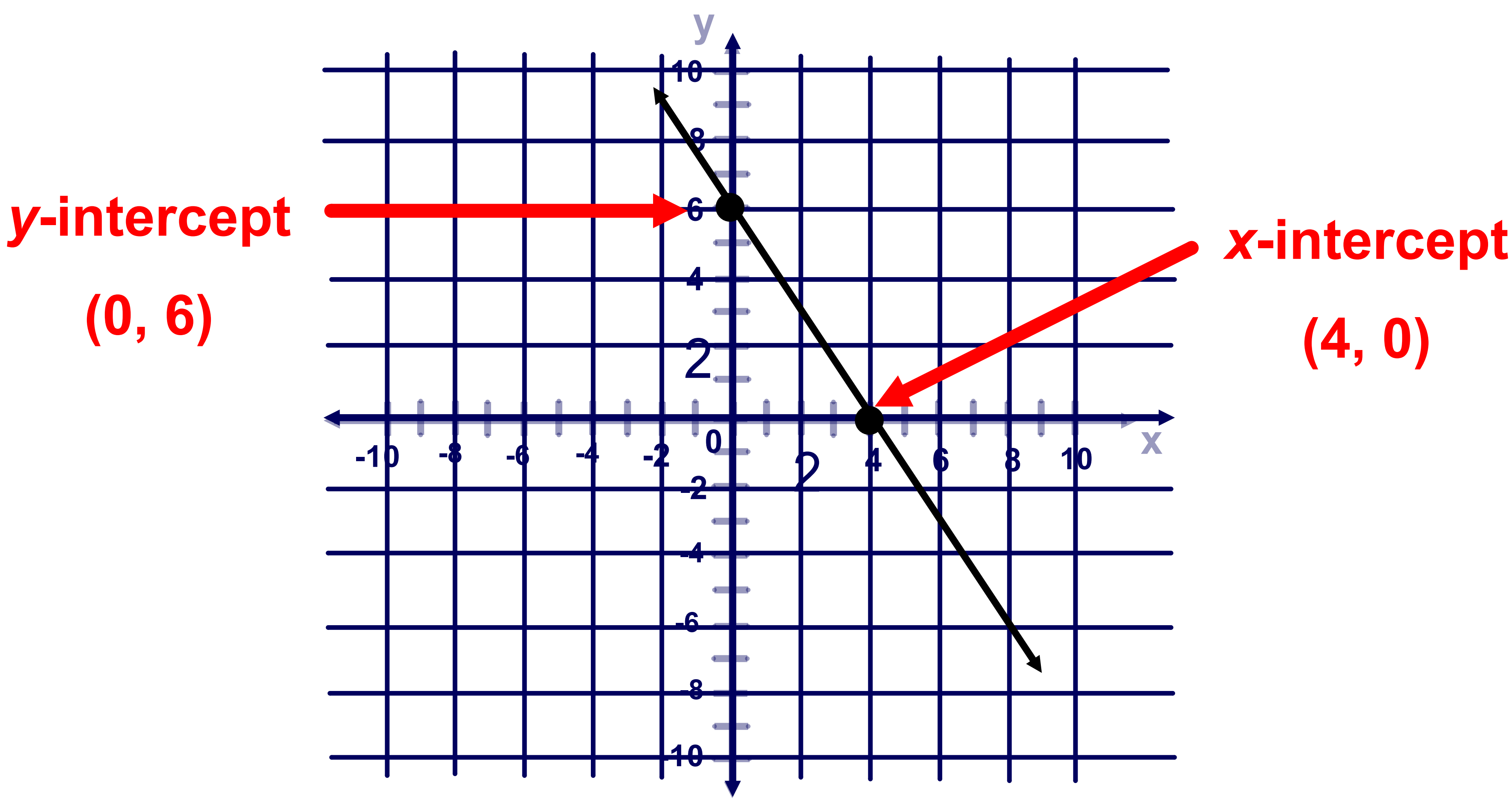
Graphing Rational Functions

Return to Table of Contents

Vocabulary Review

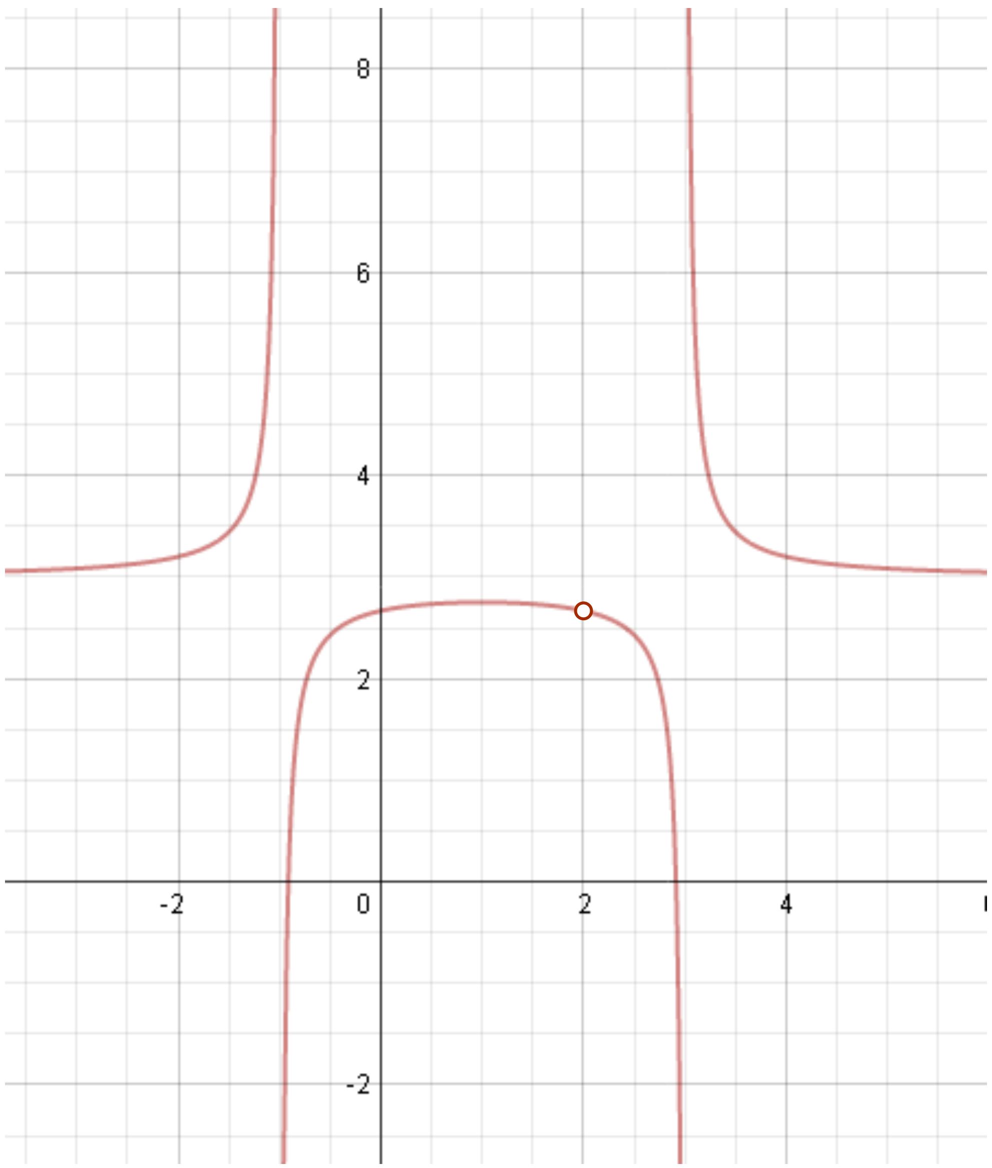
***x*-intercept**: The point where a graph intersects with the *x*-axis and the *y*-value is zero.

***y*-intercept**: The point where a graph intersects with the *y*-axis and the *x*-value is zero.



Graphs

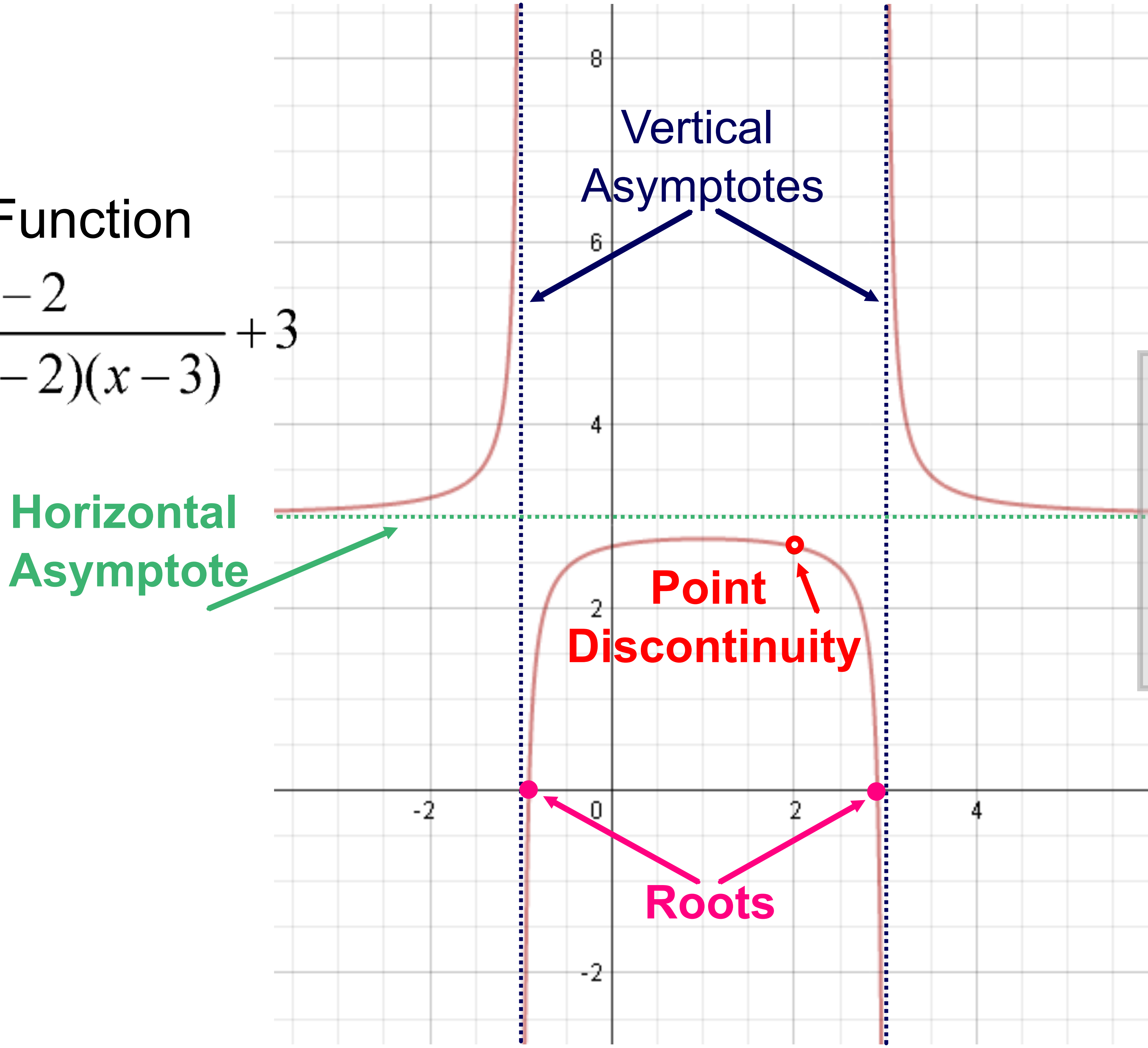
Rational Functions have unique graphs that can be explored using properties of the function itself. Here is a general example of what the graph of a rational function can look like:



Visual Vocabulary

Rational Function

$$f(x)=\frac{x-2}{(x+1)(x-2)(x-3)}+3$$



Teacher Notes

Vocabulary

Rational Function: $$f(x) = \frac{polynomial}{polynomial}$$

Roots: *x*-intercept(s) of the function;

x values for which the numerator = 0

Discontinuities: *x*-values for which the function is undefined;

Infinite discontinuity: *x*-values for which only the denominator = 0
(vertical asymptote)

Point discontinuity: *x*-values for which the numerator & denominator = 0
(hole)

Asymptote: A line that the graph continuously approaches but does not intersect

Graphing a Rational Function

Step 1: Find and graph vertical discontinuities

Step 2: Find and graph horizontal asymptotes

Step 3: Find and graph *x*- and *y*-intercepts

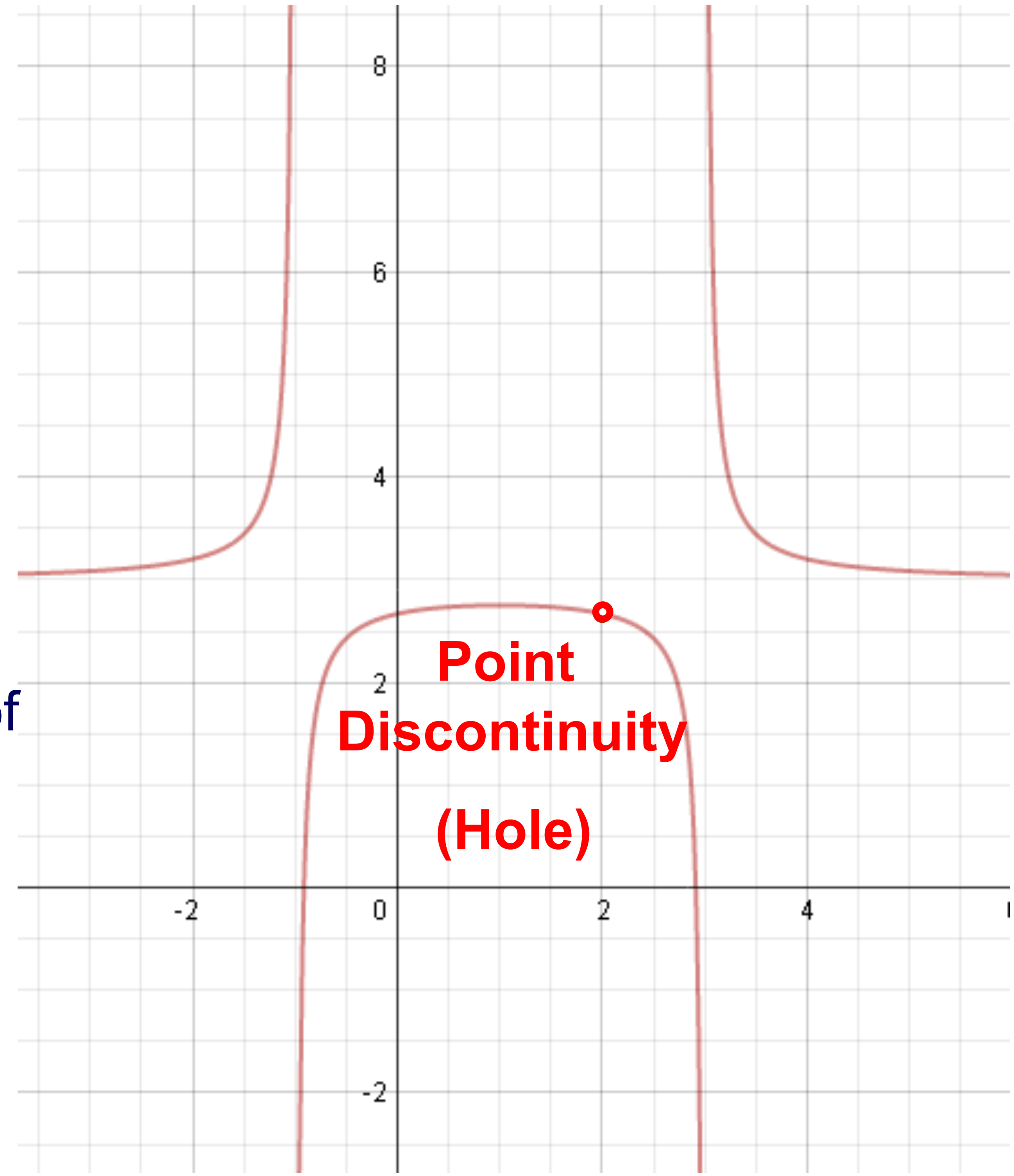
Step 4: Use a table to find values between the *x*- and *y*-intercepts

Step 5: Connect the graph

Step 1

$$f(x) = \frac{x-2}{(x+1)(x-2)(x-3)} + 3$$

A) Identify common factors from the numerator and denominator, set equal to zero and solve - Holes



x - 2 is a factor in the numerator and denominator of the rational function

$x - 2 = 0$

$x = 2$

There is a hole at $x = 2$

Step 1 Continued

$$f(x)=\frac{x-2}{(x+1)(x-2)(x-3)}+3$$

B) Set remaining denominator factors equal to zero and solve - Vertical Asymptotes

$x + 1 = 0$

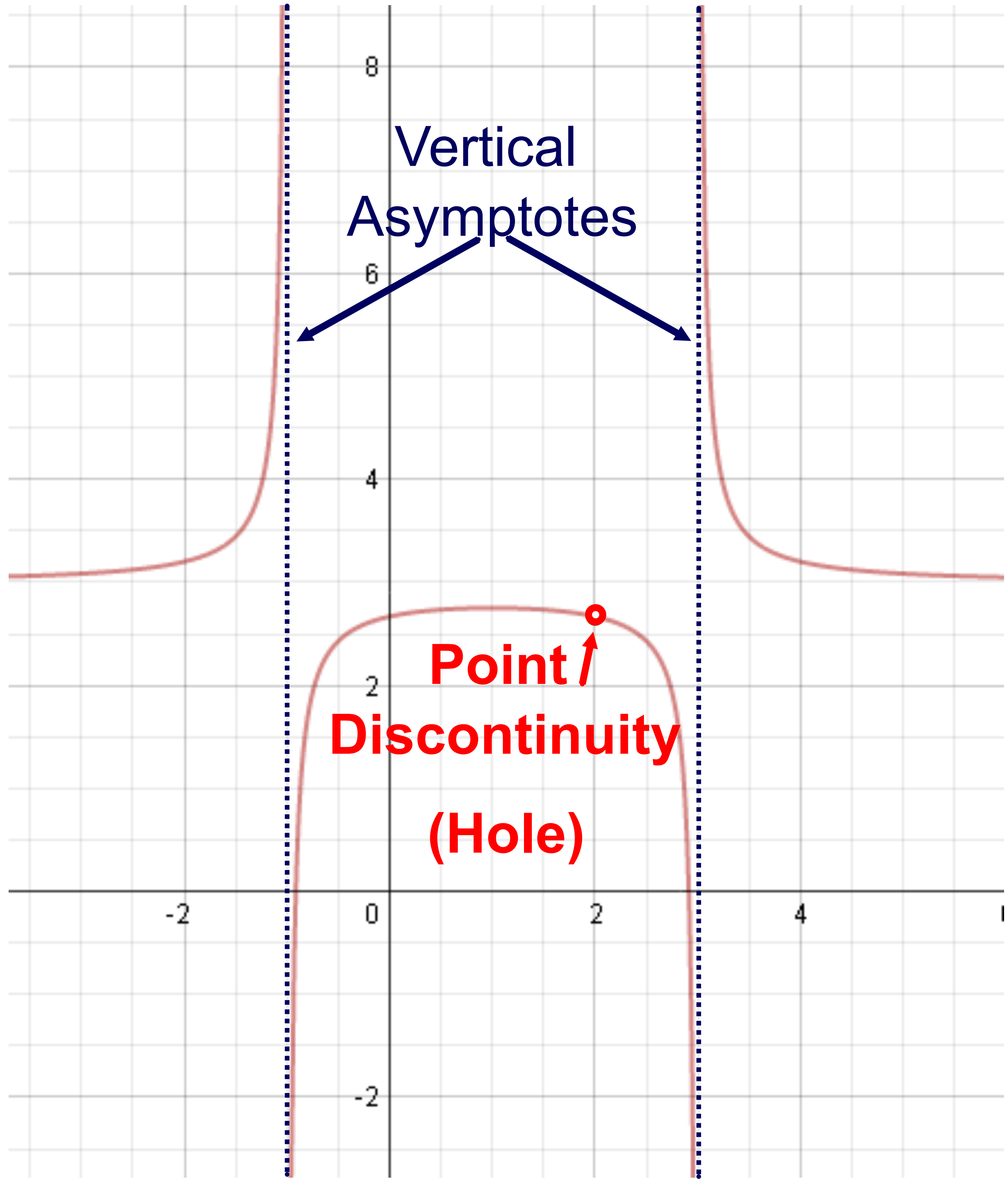
$x = -1$

Vertical Asymptote at $x = -1$

$x - 3 = 0$

$x = 3$

Vertical Asymptote at $x = 3$



Example

Find the discontinuities for the following rational function:

$$f(x) = \frac{3x+4}{(3x+4)(x+1)(x-3)}$$

A) Common Factors of numerator and denominator

$3x + 4 = 0$

Hole at $x = -\frac{4}{3}$

B) Remaining denominator factors

$x + 1 = 0$ $\quad$ $x - 3 = 0$

$x = -1$ $\quad$ $x = 3$

Vertical Asymptotes at $x = -1$ and $x = 3$

41 What are the point discontinuities of the following function:

$$f(x) = \frac{(x-1)(2x+1)}{(2x+1)(x-3)(x-1)}$$

(Choose all that apply.)

Answer

A $x = -3$

B $x = -2$

C $x = -1$

D $x = -\frac{1}{2}$

E $x = \frac{1}{2}$

F $x = 1$

G $x = 2$

H $x = 3$

42 What are the point discontinuities of the following function:

$$g(x) = \frac{x^2 + 5x}{x^3 - 9x}$$

(Choose all that apply.)

A $x = -5$

B $x = -3$

C $x = -\frac{5}{3}$

D $x = 0$

E $x = \frac{5}{3}$

F $x = 3$

G $x = 5$

H $x = 9$

Answer

43 What are the point discontinuities of the following function:

$$h(x) = \frac{x^3 - x^2 - 6x}{x^3 - 3x^2 - 10x}$$

(Choose all that apply.)

Answer

A	x = -5	E	x = 2
B	x = -3	F	x = 3
C	x = -2	G	x = 5
D	x = 0	H	x = 10

44 Find the vertical asymptotes of the following function:

$$g(x) = \frac{x^2}{x^3 - 2x}$$

(Choose all that apply.)

A $x = -3$

B $x = -2$

C $x = -\sqrt{2}$

D $x = 0$

E $x = \sqrt{2}$

F $x = 2$

G $x = 3$

H no vertical discontinuities

Answer

45 Find the vertical asymptotes of the following function:

$$f(x) = \frac{x^2 + 7x + 12}{(x-2)(x^2 + x - 12)}$$

(Choose all that apply.)

Answer

A	$x = -6$	E	$x = 2$
B	$x = -4$	F	$x = 3$
C	$x = -3$	G	$x = 4$
D	$x = -2$	H	$x = 6$

46 Discuss the discontinuities of:

$$h(x) = \frac{x}{x-1}$$

Answer

47 Discuss the discontinuities of:

$$g(x) = \frac{x+2}{(x-3)(x+2)}$$

Answer

48 Discuss the discontinuities of:

$$y = \frac{x-3}{x^2-9}$$

Answer

Notation for Holes

The point discontinuities (holes) in the graph of a rational function should be given as an ordered pair.

Once the *x*-value of the hole is found, substitute for *x* in the simplified rational expression to obtain the *y*-value.

Example

Find the holes in the graph of the following rational function:

$$g(x) = \frac{x+2}{(x-3)(x+2)}$$

Common factor of numerator and denominator:

$x + 2 = 0$

Hole at $x = -2$

Simplified expression: $$\frac{1}{(x-3)}$$

Evaluate for $x = -2$: $$\frac{1}{(-2-3)} = -\frac{1}{5}$$

The hole of this function is at (-2, -1/5)

Example

Find the holes in the graph of the following rational function:

$$h(x) = \frac{x-3}{x^2-9}$$

Common factors of numerator and denominator:

x - 3 = 0

$$h(x) = \frac{x-3}{(x-3)(x+3)}$$

Hole at x = 3

Simplified expression: ***click*** ________

Evaluate for x = 3:

click ________

The hole of this function is at ***click*** ________

49 Identify the hole(s) of the following function:

$$h(x) = \frac{x}{x-1}$$

(Choose all that apply.)

A (1, 1)

B (-1, 1)

C (1, 0)

D no holes exist

Answer

50 Identify the hole(s) of the following function:

$$h(x) = \frac{x^3 - x^2 - 6x}{x^3 - 3x^2 - 10x}$$

(Choose all that apply.)

A $(0, \frac{3}{5})$

B (0, 0)

C $(-2, \frac{5}{7})$

D there are no holes

Answer

51 Identify the hole(s) of the following function:

$$f(x)=\frac{x^2+7x+12}{(x-2)(x^2+7x+12)}$$

(Choose all that apply.)

A $(-4,-\frac{1}{6})$

B $(-3,-\frac{1}{5})$

C (3, 1)

D there are no holes

Answer

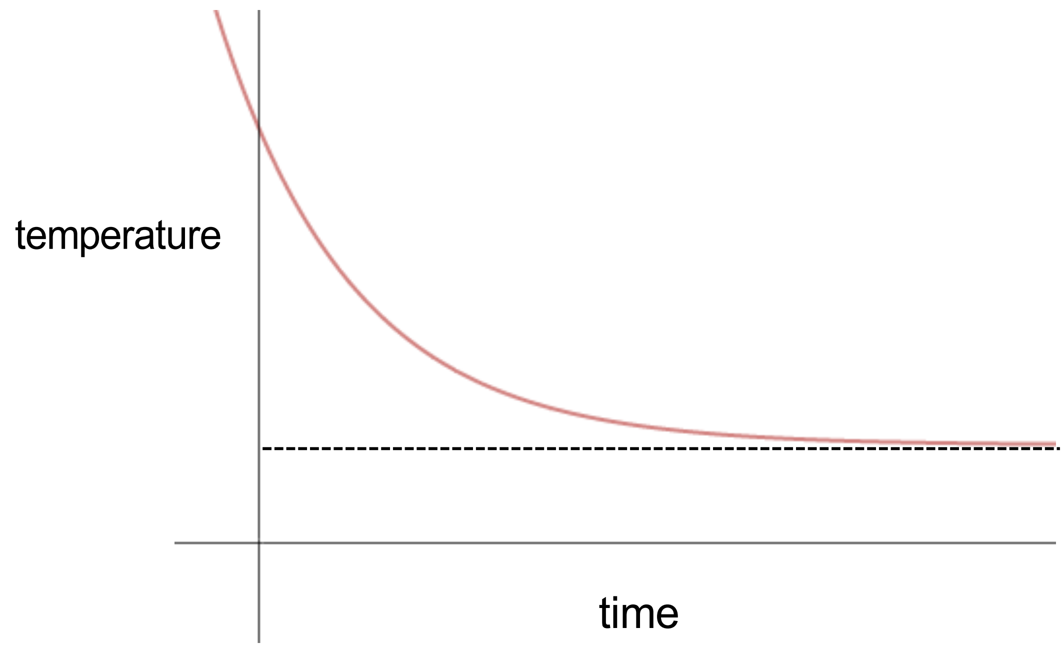
Step 2: Horizontal Asymptotes

The horizontal asymptote of a rational function is determined by comparing the degree of the numerator to the degree of the denominator.

The horizontal asymptote provides guidance for the graph's behavior as *x*-values become very large or very small. In other words, as *x* approaches ∞ or as *x* approaches $-\infty$.

Example

Think of a cup of boiling water left on a table to cool. If you graph the temperature for a period of time, what would be considered the horizontal asymptote and why?



Horizontal Asymptote = Room Temperature

The limiting factor is the room temperature. The water is not able to co below room temperature, so the graph will have a horizontalasymptot

Horizontal Asymptotes

To find the horizontal asymptotes of a function, compare the degree of the numerator to the degree of the denominator.

n = degree of numerator   m = degree of denominator

Use the following rules:

n > m	n = m	n < m
If the numerator has a higher degree then there is no horizontal asymptote.	If the degree is the same then the horizontal asymptote is the line $y = \frac{a}{b}$.	If the denominator has a higher degree then $y = 0$ is the horizontal asymptote

Teacher Notes

Degree

Recall from Algebra I

The **degree** of a polynomial is the term containing the variable raised to the highest exponent.

Remember: A **constant** has a degree of 0. A variable with no exponent has a degree of 1.

For Example:

What is the degree of the polynomial $-6x3 + 2x$?

First Term is $-6x3$: x has a power of 3, so the degree is 3

Second Term is $2x$: x has a power of 1, so the degree is 1

The degree of the polynomial is 3.

Example

Decide if the following function has a horizontal asymptote. If so, find the equation of the asymptote.

$$y = \frac{6x^4}{x^3 + 2x - 7}$$

Degree of Numerator = 4

Degree of Denominator = 3

n > m Therefore, no horizontal asymptote

n > m	n = m	n < m
If the numerator has a higher degree, there is **no** horizontal asymptote.	If the degree is the same, then the horizontal asymptote is the line $y = \frac{a}{b}$.	**If the denominator has a** higher degree, then **y = 0** is the horizontal asymptote

Example

Decide if the following function has a horizontal asymptote. If so, find the equation of the asymptote.

$$y=\frac{6x^5-4x^3+2x}{7x^5-12x^4+3x-20}$$

Degree of Numerator = 5

Degree of Denominator = 5

n = m Therefore, horizontal asymptote is the line $y=\frac{a}{b}$ where

a is the leading coefficient in the numerator

and

b is the leading coefficient in the denominator

The horizontal asymptote is $y=\frac{6}{7}$

Horizontal Asymptotes

Try these: Decide if the following functions have horizontal asymptotes. If so, find the equation of the asymptote.

a. $$y = \frac{x^4}{x^2 - 7}$$

b. $$y = \frac{1}{x^3 + 2x - 7}$$

Teacher Notes

52 Given f(x), which of the choices best describes the horizontal asymptote?

$$f(x) = \frac{9x^5 - 4x^3 + 2x}{3x^5 - 12x^4 + 3x - 20}$$

Answer

A f(x) has no horizontal asymptote

B $y = 0$

C $y = 3$

D $y = \frac{1}{3}$

53 Given f(*x*), which of the choices best describes the horizontal asymptote?

$$f(x) = \frac{7x^4 - 4x^3 + 2x}{3x^5 - 12x^4 + 3x - 20}$$

A f(x) has no horizontal asymptote.

B $y = 0$

C $y = \frac{7}{3}$

D $y = \frac{3}{7}$

Answer

54 Given f(x), which of the choices best describes the horizontal asymptote?

$$f(x) = \frac{8x^6 - 4x^3 + 2x}{4x^5 - 12x^4 + 3x - 20}$$

A f(x) has no horizontal asymptote.

B $y = 0$

C $y = 2$

D $y = 1/2$

Answer

55 Given f(x), which of the choices best describes the horizontal asymptote?

$$f(x) = \frac{-8x^6 - 4x^3 + 2x}{-4x^6 - 12x^4 + 3x - 20}$$

Answer

A f(x) has no horizontal asymptote.

B $y = 0$

C $y = -2$

D $y = 2$

Step 3: Intercepts

x-intercepts

The ***x*-intercept(s)** occur when *y* = 0, or where the numerator equals zero.

Set the numerator equal to zero and solve to find the *x*-intercepts.

Intercepts should be named as ordered pairs.

****Remember, if this value makes the denominator zero as well, there is a point discontinuity (a hole)****

Intercepts

y-intercepts

The ***y*-intercepts** occur where x is equal to zero.

Substitute zero for all x's and solve to find the y-intercepts.

Intercepts should be named as ordered pairs.

Intercepts

Find the *x* and *y*-intercepts of the following function:

$$f(x) = \frac{x-4}{(x+1)(x-2)(x-3)}$$

x-intercept(s)	*y*-intercept(s)
Set the numerator equal to zero and solve to find the *x*-intercepts.	Evaluate for x = 0 to find the y-intercepts.
click	***click***

56 Identify the y-intercept of

$$f(x) = \frac{x}{x-1}$$

Answer

57 Identify the y-intercept of

$$f(x) = \frac{x+1}{x-1}$$

Answer

58 Find the y-intercept of

$$f(x) = \frac{x+2}{x^2-9}$$

Answer

59 What are the *y*-intercepts for the following function?

$$f(x) = \frac{x+2}{x^2}$$

(Choose all that apply.)

Answer

A (0, -6)

B (0, -3)

C (0, 0)

D (0, 3)

E (0, 6)

F There are no real intercepts

60 Find any *x*-intercept(s) of:

$$h(x) = \frac{x}{x-1}$$

A (-3, 0)

B (-1, 0)

C (0, 0)

D (1, 0)

E (3, 0)

F There are no real intercepts

Answer

61 Find all *x*-intercept(s) of:

$$g(x) = \frac{x+2}{(x-3)(x+2)}$$

A (-3, 0)

B (-2, 0)

C (0, 0)

D (2, 0)

E (3, 0)

F There are no real intercepts

Answer

62 Identify all *x*-intercept(s) of:

$$y = \frac{(x-3)(x^2-4)}{(x^2-9)}$$

Answer

A (-3, 0)

B (-2, 0)

C (0, 0)

D (2, 0)

E (3, 0)

F There are no real intercepts

63 Choose all *x*-intercept(s) of:

$$y = \frac{(x^3 - 9x)}{(x^2 - 4)}$$

A (-3, 0)

B (-2, 0)

C (0, 0)

D (2, 0)

E (3, 0)

F There are no real intercepts

Answer

Step 4: Table

Graphs of rational functions contain curves, and additional points are needed to ensure the shape of the graph.

Once all discontinuities, asymptotes and intercepts are graphed, additional points can be found by creating a table of values.

To create an accurate graph, it is good practice to choose *x*-values near the intercepts and vertical asymptotes.

Example

Graph: $f(x) = \frac{x^2 - x - 6}{x^2 - 4}$

Use factoring to help identify discontinuities and intercepts:

$$f(x) = \frac{(x-3)(x+2)}{(x-2)(x+2)}$$

Step 1: Discontinuities

A) Common Factors of numerator and denominator

$x + 2 = 0$

Hole at $x = -2$

$$(-2, \frac{5}{4})$$

B) Remaining denominator factors

$x - 2 = 0$

$x = 2$

Vertical Asymptote at $x = 2$

Example Continued

Step 2: Horizontal Asymptotes

$$f(x) = \frac{x^2 - x - 6}{x^2 - 4}$$

Check the degree of numerator and denominator.

Since n = m, the asymptote is $y = \frac{a}{b}$

The asymptote for this graph is $y = 1$

Example Continued

Step 3: *x* and *y*-intercepts

$$f(x) = \frac{(x-3)(x+2)}{(x-2)(x+2)}$$

x-intercepts	*y*-intercept(s)
Set the numerator equal to zero and solve to find the *x*-intercepts. (Exclude factors that are common to numerator and denominator.)	Evaluate for *x* = 0 to find the *y*-intercepts.
$x - 3 = 0$	$f(0) = \frac{0^2 - 0 - 6}{0^2 - 4} = \frac{-6}{-4} = \frac{3}{2}$
$x = 3$	$(0, \frac{3}{2})$
(3, 0)	

Example Continued

Step 4: Create a table of additional ordered pairs.

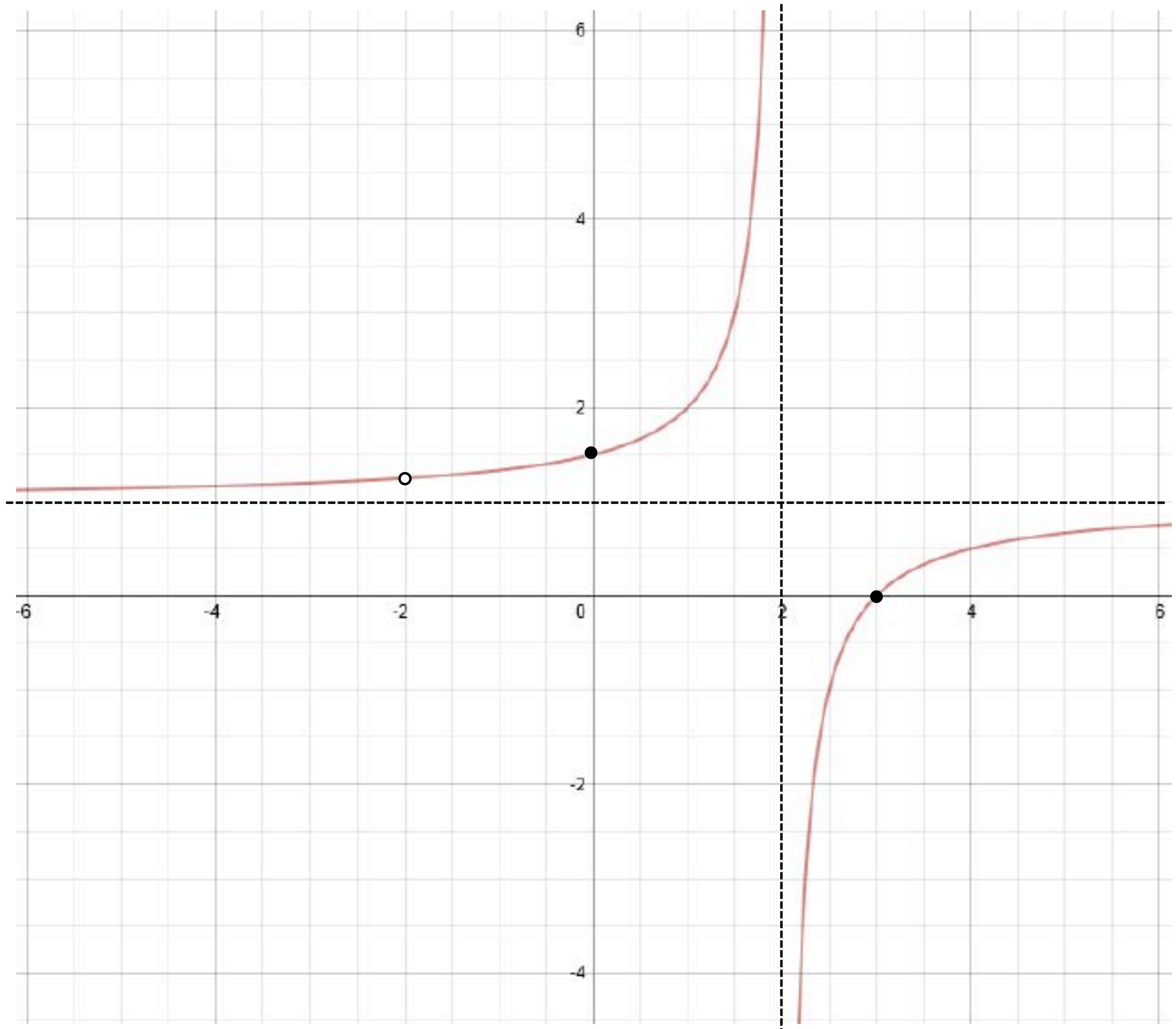
Choose values for *x* on either side of vertical asymptotes and *x*-intercepts.

$$f(x)=\frac{x^2-x-6}{x^2-4}$$

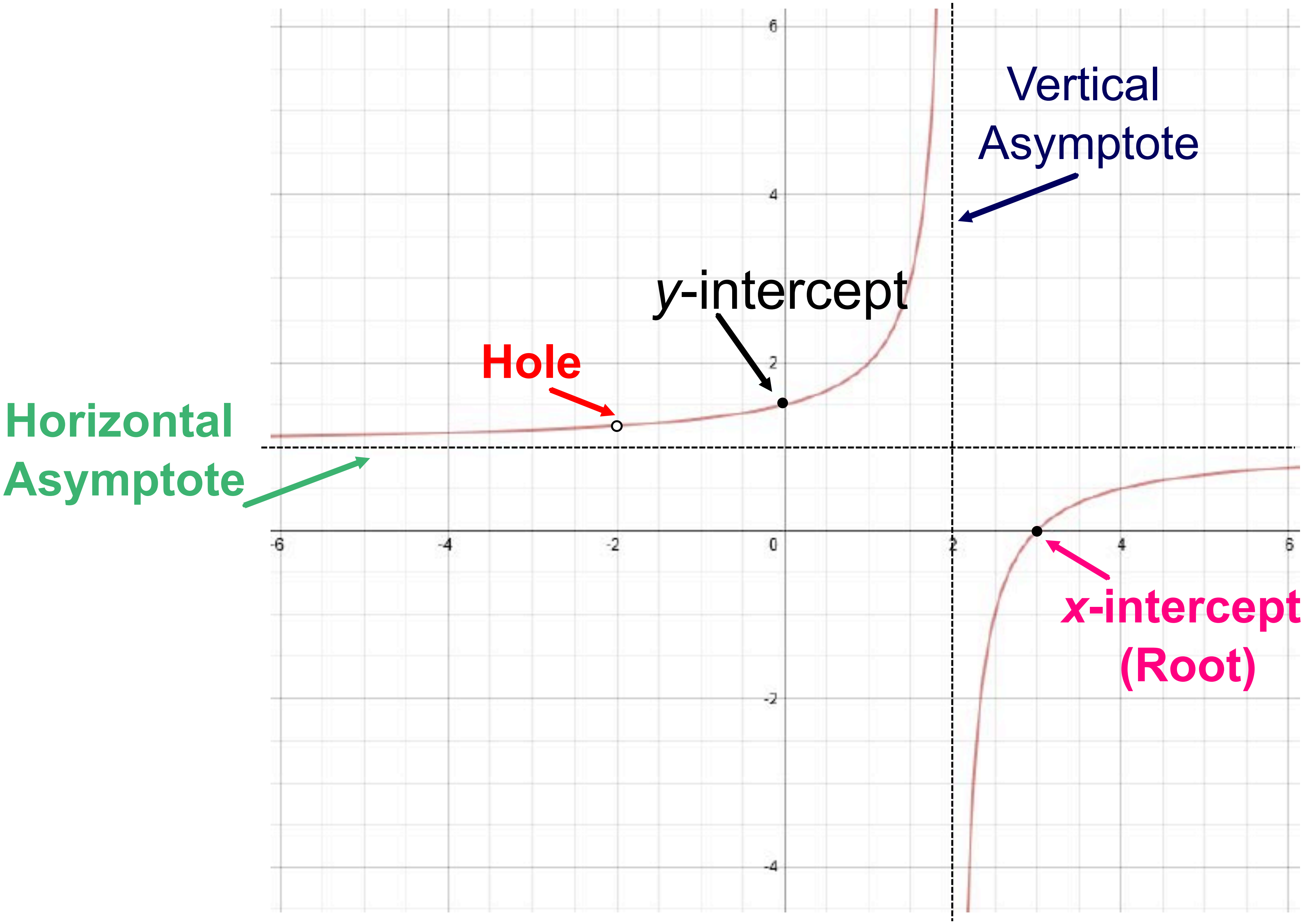
x	y
-4	1.17
-3	1.2
-1	1.3
0	1.5
1	2
3	0
4	0.5
5	0.67

Example Continued

Step 5: Connect the points with a smoothe curve. $f(x)=\dfrac{x^2-x-6}{x^2-4}$



Graph Components



64 What is the first step to use when graphing rational functions?

A Finding the intercepts

B Finding the horizontal asymptote

C Creating a table of values

D Creating the graph by connecting all previously found components

E Finding the discontinuities

Answer

65 The correct notation for a hole in a rational function is:

A $x = 2$

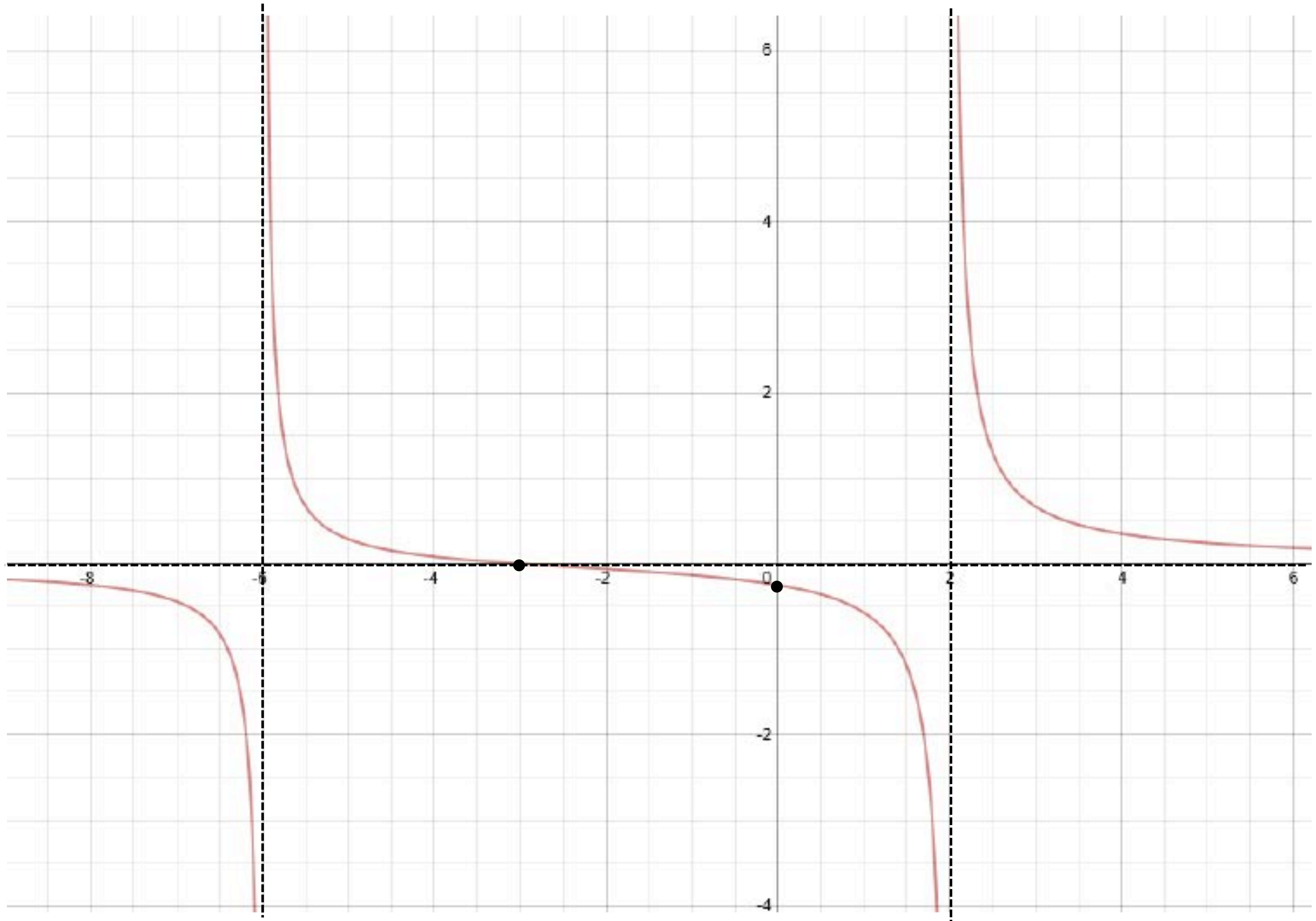
B $y = 2$

C (2, 5)

D There is no correct notation.

Answer

66 Which function corresponds to the following graph?



A $f(x)=\dfrac{x+3}{x^2+4x-12}$

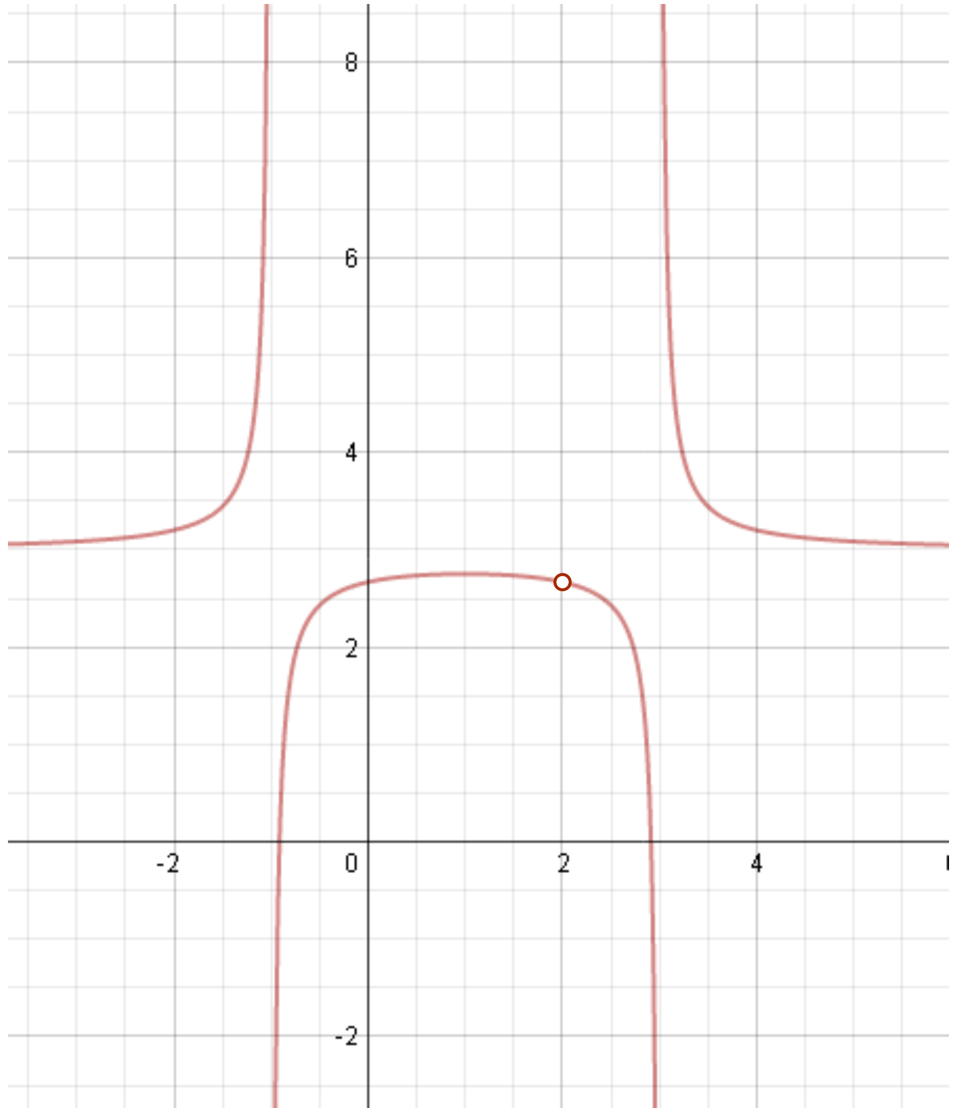
B $f(x)=\dfrac{1}{x^2+4x-12}$

C $f(x)=\dfrac{x+3}{x^2-4x+12}$

D $f(x)=\dfrac{1}{x^2-4x+12}$

Answer

67 Which function corresponds to the following graph?



A $f(x)=\dfrac{x-2}{(x-1)(x-2)(x+3)}$

B $f(x)=\dfrac{1}{(x+1)(x-3)}+3$

C $f(x)=\dfrac{x-2}{(x+1)(x-2)(x-3)}$

D $f(x)=\dfrac{x-2}{(x+1)(x-2)(x-3)}+3$

Answer

Graph 1

Now, let's put it all together.

Step 1: Find and graph vertical discontinuities

$$f(x) = \frac{x+3}{x^2+4x-12}$$

Teacher Notes

Graph 1

Step 2: Find and graph horizontal asymptotes

$$f(x) = \frac{x+3}{x^2+4x-12}$$

Teacher Notes

Graph 1

Step 3: Find and graph *x*- and *y*-intercepts

$$f(x) = \frac{x+3}{x^2+4x-12}$$

Teacher Notes

Graph 1

Step 4: Use a table to find values between the *x*- and *y*-intercepts

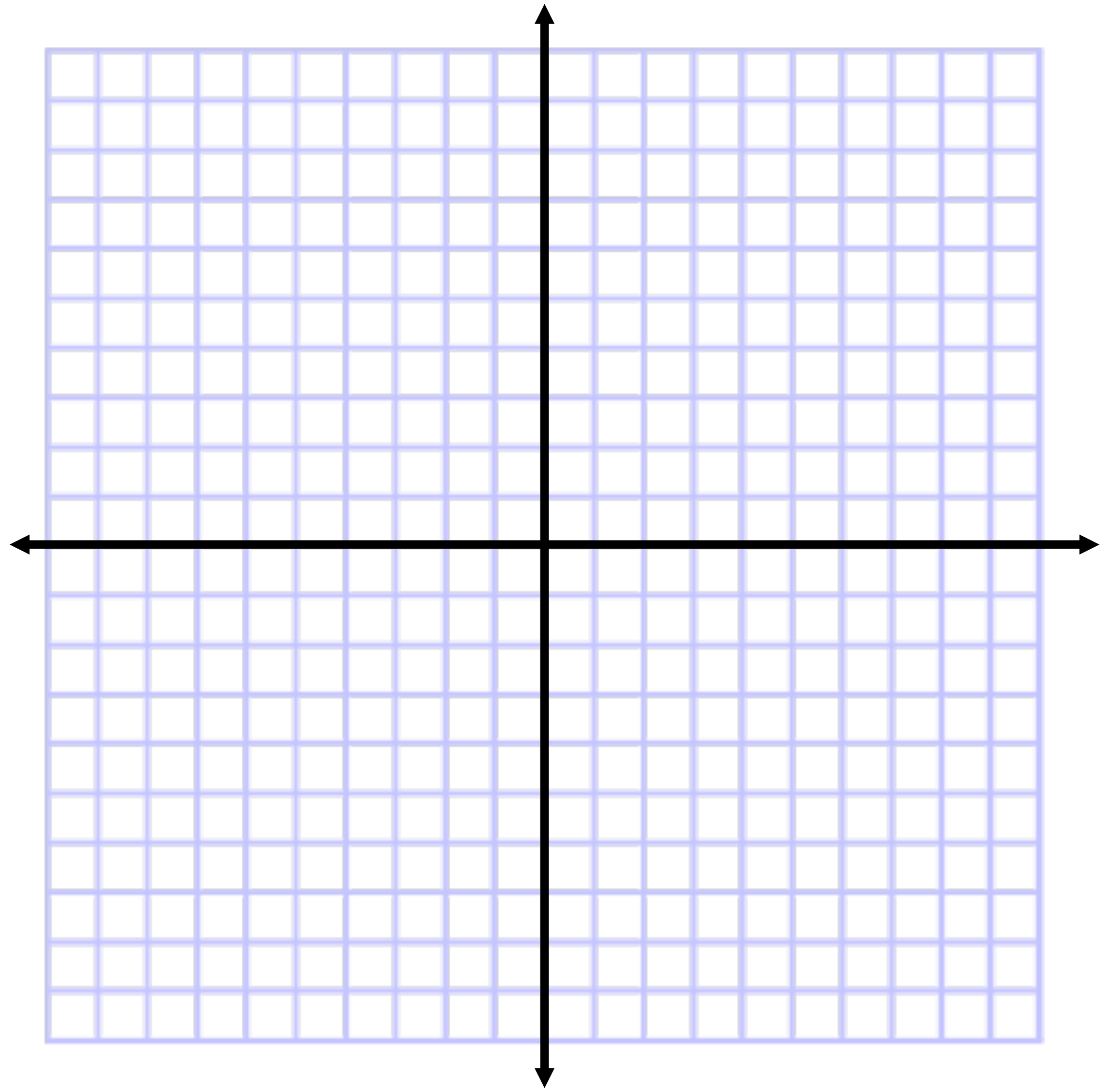
$$f(x) = \frac{x+3}{x^2+4x-12}$$

Teacher Notes

Graph 1

Step 5: Connect the graph

$$f(x) = \frac{x+3}{x^2+4x-12}$$



Teacher Notes

Graph 2

Try another example.

Step 1: Find and graph vertical discontinuities

$$f(x) = \frac{3}{x+1} + 2$$

Teacher Notes

Graph 2

Step 2: Find and graph horizontal asymptotes

$$f(x) = \frac{3}{x+1} + 2$$

Teacher Notes

Graph 2

Step 3: Find and graph *x*- and *y*-intercepts

$$f\left(x\right)=\frac{3}{x+1}+2$$

Teacher Notes

Graph 2

Step 4: Use a table to find values between the *x*- and *y*-intercepts

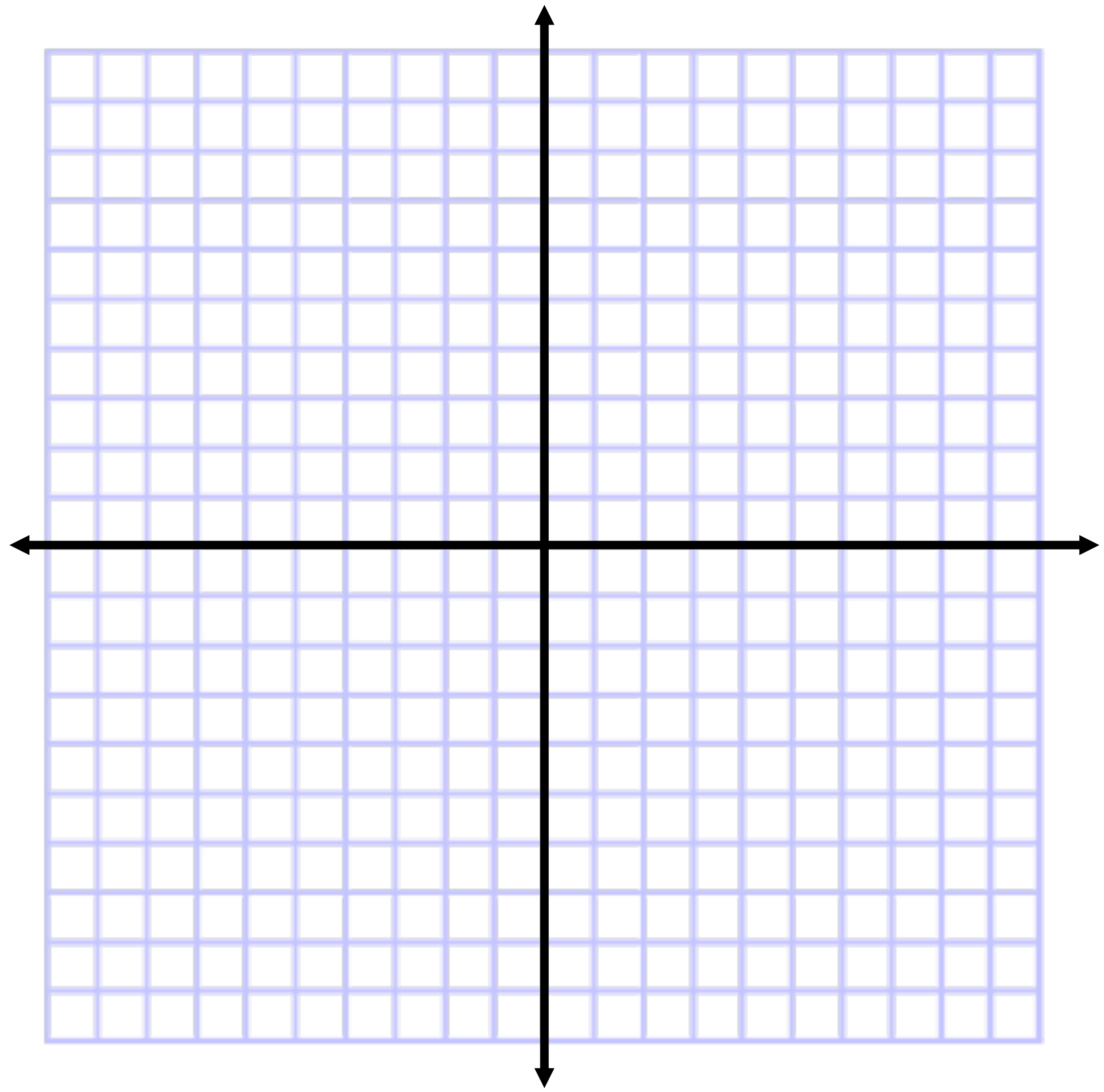
$$f(x) = \frac{3}{x+1} + 2$$

Teacher Notes

Graph 2

Step 5: Connect the graph

$$f\left(x\right)=\frac{3}{x+1}+2$$



Teacher Notes

Graph 3

Step 1: Find and graph vertical discontinuities

$$f(x)=\frac{x^3-9x}{3x^2-6x-9}$$

Teacher Notes

Graph 3

Step 2: Find and graph horizontal asymptotes

$$f(x) = \frac{x^3 - 9x}{3x^2 - 6x - 9}$$

Teacher Notes

Graph 3

Step 3: Find and graph *x*- and *y*-intercepts

$$f(x) = \frac{x^3 - 9x}{3x^2 - 6x - 9}$$

Teacher Notes

Graph 3

Step 4: Use a table to find values between the *x*- and *y*-intercepts

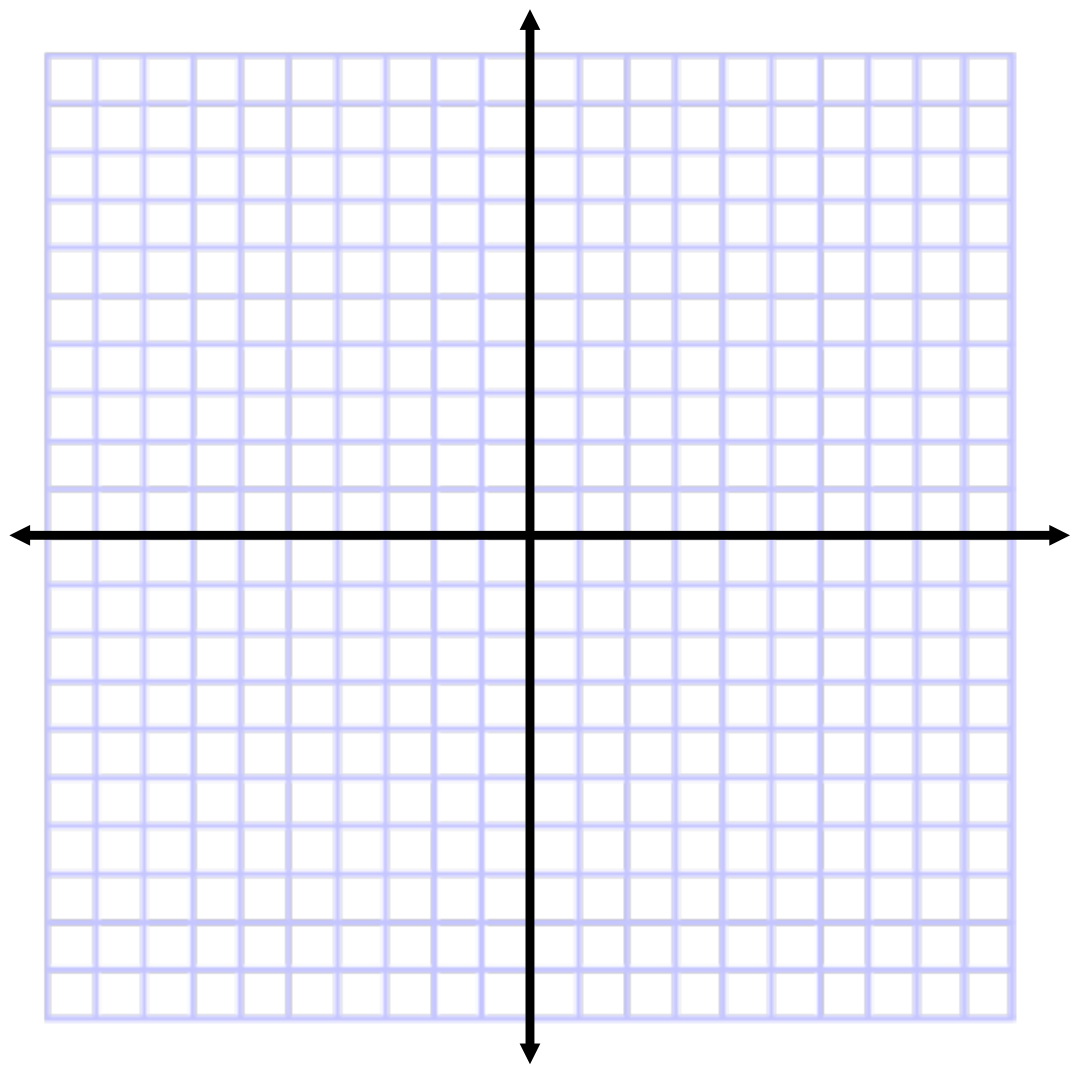
$$f(x)=\frac{x^3-9x}{3x^2-6x-9}$$

Teacher Notes

Graph 3

Step 5: Connect the graph

$$f(x)=\frac{x^3-9x}{3x^2-6x-9}$$



Teacher Notes