Logarithms and Exponentials

Section 1 Logarithms

The mathematics of logarithms and exponentials occurs naturally in many branches of science. It is very important in solving problems related to growth and decay. The growth and decay may be that of a plant or a population, a crystalline structure or money in the bank. Therefore we need to have some understanding of the way in which logs and exponentials work.

Definition: If x and b are positive numbers and $b \neq 1$ then the logarithm of x to the base b is the power to which b must be raised to equal x. It is written $\log_b x$. In algebraic terms this means that

$$\begin{aligned} \text{if } y &= \log_b x \quad \text{then} \\ x &= b^y \end{aligned}$$

The formula $y = \log_b x$ is said to be written in logarithmic form and $x = b^y$ is said to be written in exponential form. In working with these problems it is most important to remember that $y = \log_b x$ and $x = b^y$ are equivalent statements.

Example 1 : If $\log_4 x = 2$ then

$$\begin{aligned} x &= 4^2 \\ x &= 16 \end{aligned}$$

Example 2 : We have $25 = 5^2$. Then $\log_5 25 = 2$.

Example 3 : If $\log_9 x = \frac{1}{2}$ then

$$\begin{aligned} x &= 9^{\frac{1}{2}} \\ x &= \sqrt{9} \\ x &= 3 \end{aligned}$$

Example 4 : If $\log_2 \frac{y}{3} = 4$ then

$$\begin{aligned} \frac{y}{3} &= 2^4 \\ \frac{y}{3} &= 16 \\ y &= 16 \times 3 \end{aligned}$$

Logs have some very useful properties which follow from their definition and the equivalence of the logarithmic form and exponential form. Some useful properties are as follows:

$$\begin{aligned}
\log_b mn &= \log_b m + \log_b n \\
\log_b \frac{m}{n} &= \log_b m - \log_b n \\
\log_b m^a &= a \log_b m \\
\log_b m &= \log_b n \quad \text{if and only if} \quad m = n
\end{aligned}$$

Example 1 :

$$\begin{aligned}
\log_b \frac{xy}{z} &= \log_b xy - \log_b z \\
&= \log_b x + \log_b y - \log_b z
\end{aligned}$$

Example 2 :

$$\begin{aligned}
\log_5 5^p &= p \log_5 5 \\
&= p \times 1 \\
&= p
\end{aligned}$$

Example 3 :

$$\begin{aligned}
\log_2 (8x)^{\frac{1}{3}} &= \frac{1}{3} \log_2 8x \\
&= \frac{1}{3} [\log_2 8 + \log_2 x] \\
&= \frac{1}{3} [3 + \log_2 x] \\
&= 1 + \frac{1}{3} \log_2 x
\end{aligned}$$

Example 4 : Find x if

$$2 \log_b 5 + \frac{1}{2} \log_b 9 - \log_b 3 = \log_b x$$

$$\begin{aligned}
\log_b 5^2 + \log_b 9^{\frac{1}{2}} - log_b 3 &= \log_b x \\
\log_b 25 + \log_b 3 - \log_b 3 &= \log_b x \\
\log_b 25 &= \log_b x \\
x &= 25
\end{aligned}$$

Example 5 :

$$\begin{aligned}
\log_2 \frac{8x^3}{2y} &= \log_2 8x^3 - \log_2 2y \\
&= \log_2 8 + \log_2 x^3 - [\log_2 2 + \log_2 y] \\
&= 3 + 3 \log_2 x - [1 + \log_2 y] \\
&= 3 + 3 \log_2 x - 1 - \log_2 y \\
&= 2 + 3 \log_2 x - \log_2 y
\end{aligned}$$

Ex

1. Write the following in exponential form:

   (a) $\log_3 x = 9$

   (b) $\log_2 8 = x$

   (c) $\log_3 27 = x$

   (d) $\log_4 x = 3$

   (e) $\log_2 y = 5$

   (f) $\log_5 y = 2$

2. Write the following in logarithm form:

   (a) $y = 3^4$

   (b) $27 = 3^x$

   (c) $m = 4^2$

   (d) $y = 3^5$

   (e) $32 = x^5$

   (f) $64 = 4^x$

3. Solve the following:

   (a) $\log_3 x = 4$

   (b) $\log_m 81 = 4$

   (c) $\log_x 1000 = 3$

   (d) $\log_2 \frac{x}{2} = 5$

   (e) $\log_3 y = 5$

   (f) $\log_2 4x = 5$

ses.

1. Use the logarithm laws to simplify the following:

   (a) $\log_2 xy - \log_2 x^2$

   (b) $\log_2 \frac{8x^2}{y} + \log_2 2xy$

   (c) $\log_3 9xy^2 - \log_3 27xy$

   (d) $\log_4 (xy)^3 - \log_4 xy$

   (e) $\log_3 9x^4 - \log_3 (3x)^2$

2. Find x if:

   (a) $2\log_b 4 + \log_b 5 - \log_b 10 = \log_b x$

   (b) $\log_b 30 - \log_b 5^2 = \log_b x$

   (c) $\log_b 8 + \log_b x^2 = \log_b x$

   (d) $\log_b (x+2) - \log_b 4 = \log_b 3x$

   (e) $\log_b (x-1) + \log_b 3 = \log_b x$