Zeros and Roots of a Polynomial Function

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Real Zeros of Polynomial Functions

For a function $f(x)$ and a real number a, if $f(a) = 0$, the following statements are equivalent:

$x = a$ is a zero of the function $f(x)$.

$x = a$ is a solution of the equation $f(x) = 0$.

$(x - a)$ is a factor of the function $f(x)$.

$(a, 0)$ is an x-intercept of the graph of $f(x)$.

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x) = 0$ has n zeros including multiples and imaginary zeros.

An imaginary zero occurs when the solution to $f(x) = 0$ contains **complex numbers**. Imaginary zeros are not seen on the graph.

Complex Numbers

Complex numbers will be studied in detail in the Radicals Unit. But in order to fully understand polynomial functions, we need to know a little bit about complex numbers.

Up until now, we have learned that there is no real number, x, such that $x^2 = -1$. However, there is such a number, known as the **imaginary unit, *i*,** which satisfies this equation and is defined as $i = \sqrt{-1}$.

The set of complex numbers is the set of numbers of the form $\boldsymbol{a + bi}$, where a and b are real numbers.

When $a = 0$, bi is called a **pure imaginary number**.

The square root of any negative number is a complex number.

For example, find a solution for $x^2 = -9$:

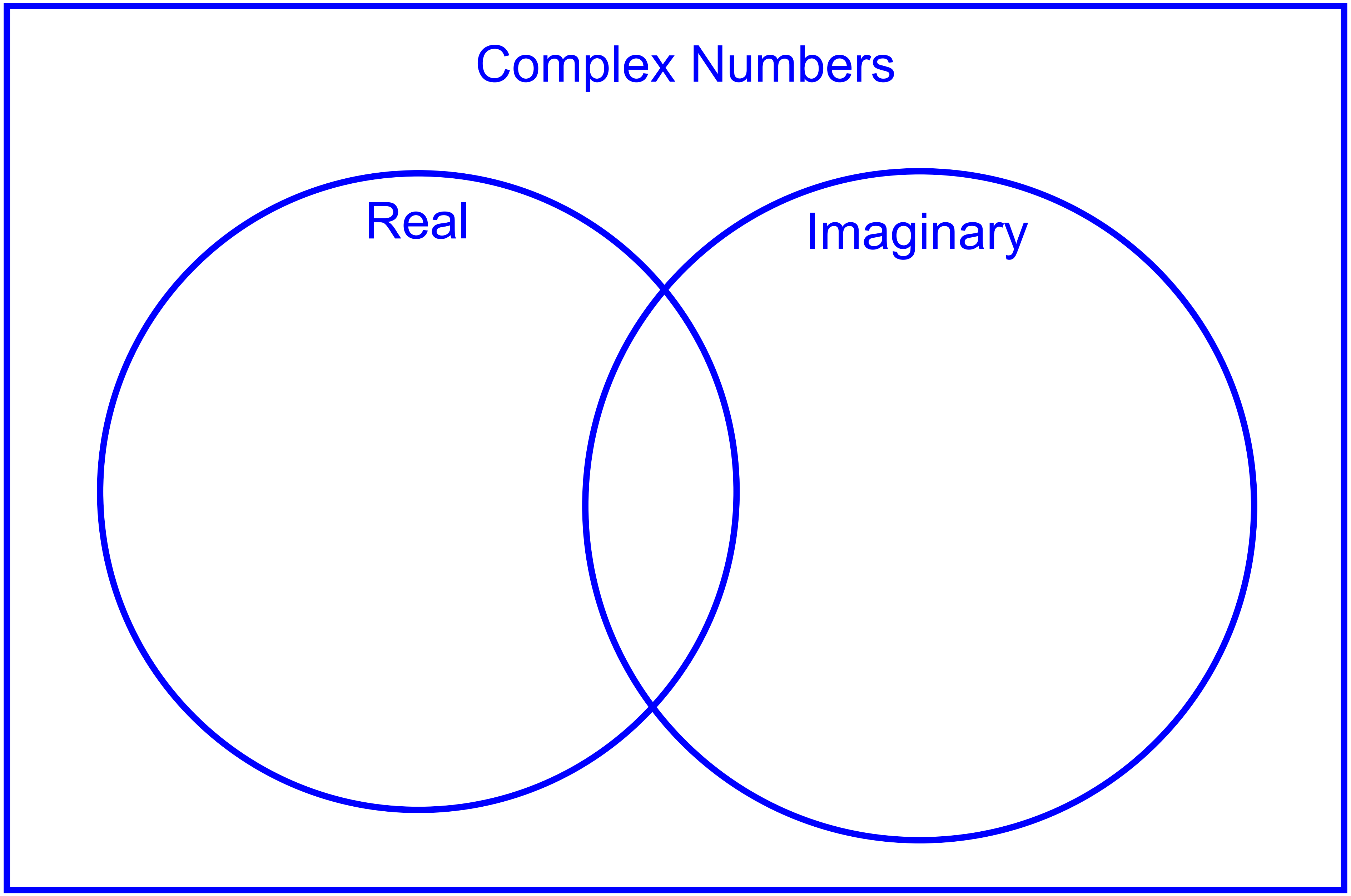
$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

$$x = \pm\sqrt{9(-1)} = \sqrt{9}\sqrt{-1}$$

$$x = \pm 3i$$

Drag each number to the correct place in the diagram.



9+6i

3i

2/3

-11

2 - 4i

$\sqrt{5}$

-0.765

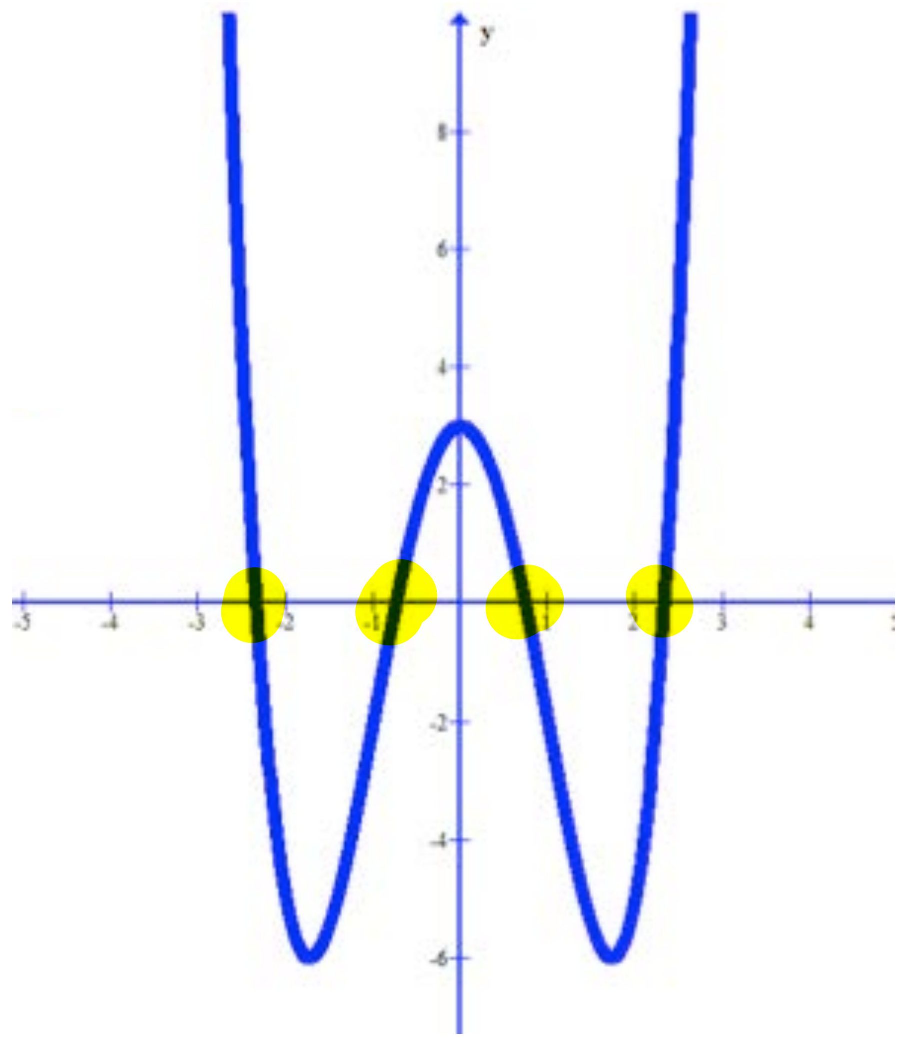
$\sqrt{-8}$

Teacher Notes

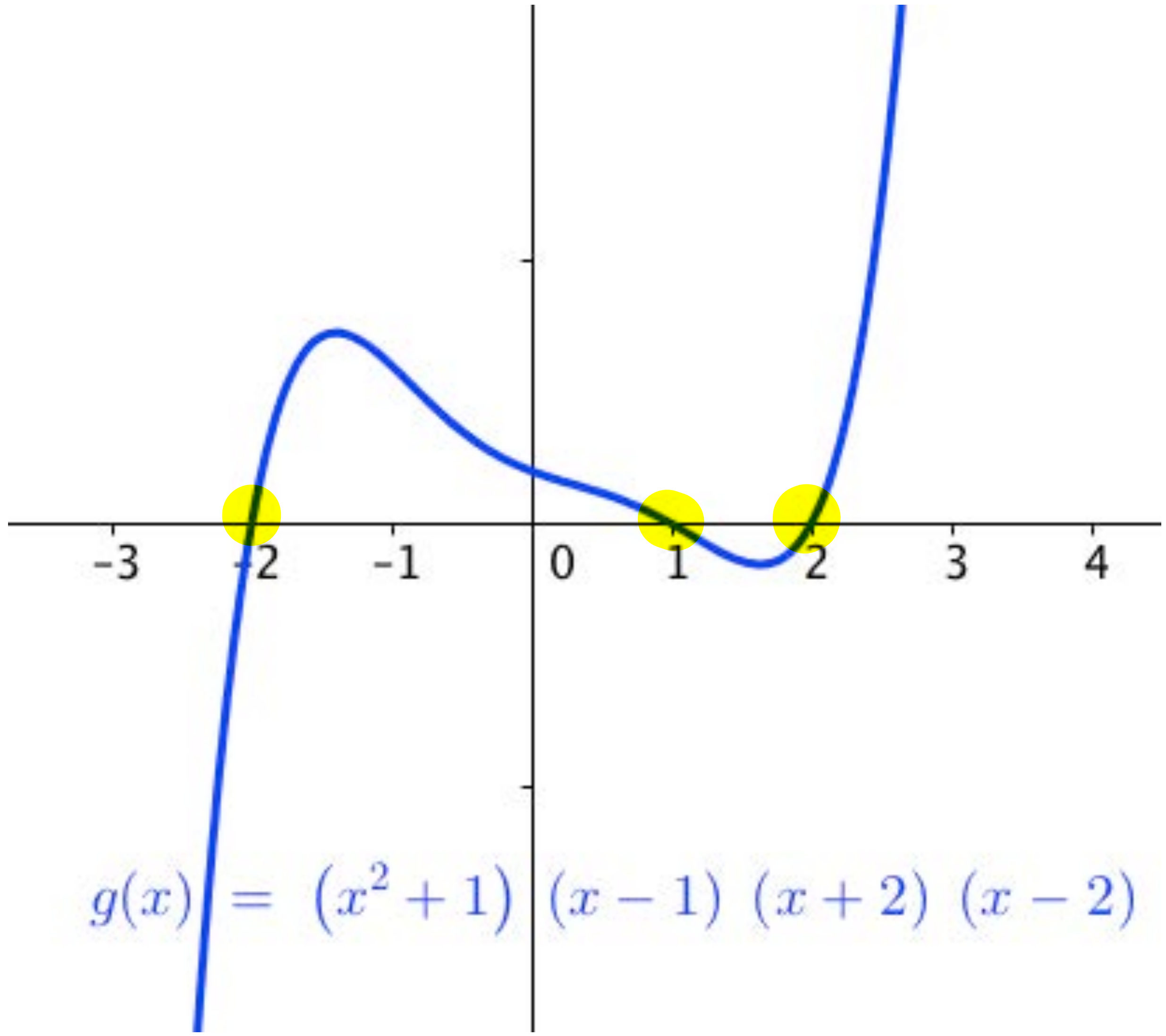
The number of the **zeros** of a polynomial, both real and imaginary, is equal to the **degree** of the polynomial.

This is the graph of a polynomial with degree 4. It has four unique zeros: -2.25, -.75, .75, 2.25

Since there are 4 real zeros, there are no imaginary zeros.
(4 in total - 4 real = 0 imaginary)



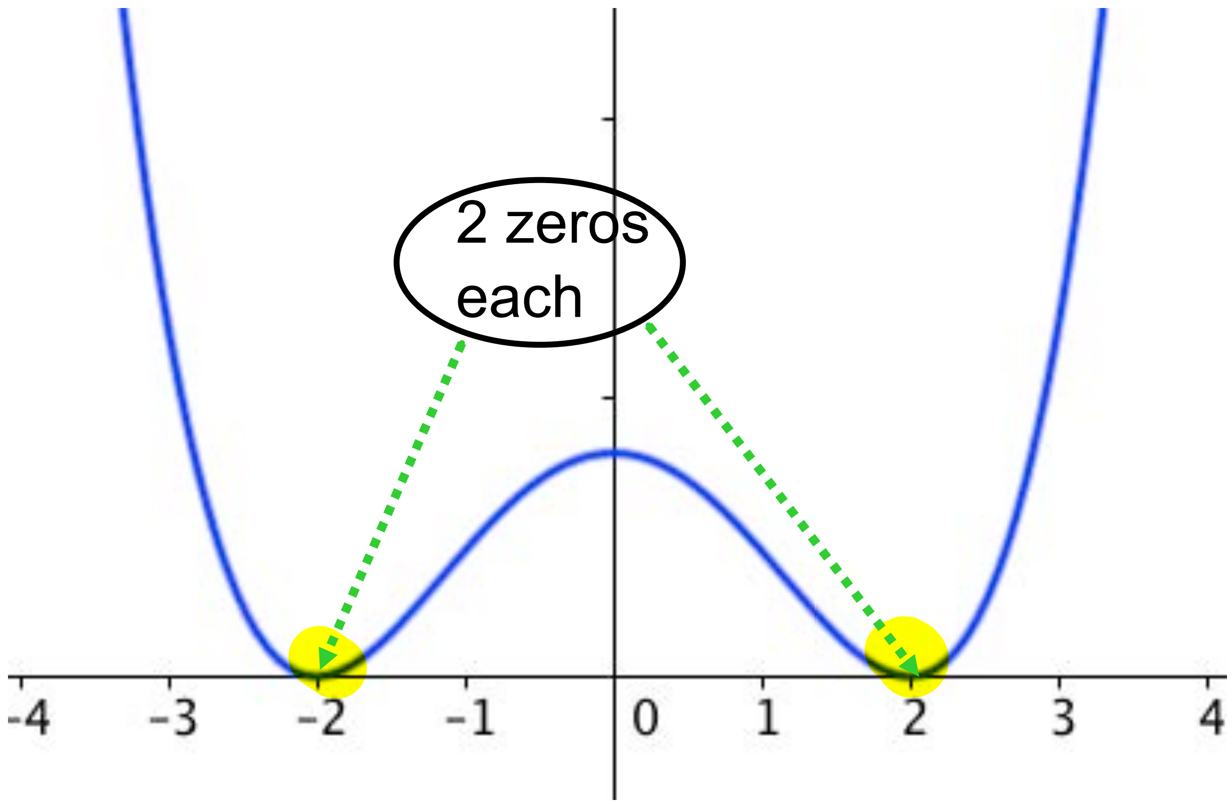
This 5th degree polynomial has 5 zeros, but only 3 of them are real. Therefore, there must be two imaginary.



(How do we know that this is a 5th degree polynomial?)

Note: imaginary roots always come in **pairs**: if $a + bi$ is a root, then a - bi is also a root. (These are called conjugates - more on that in later units.)

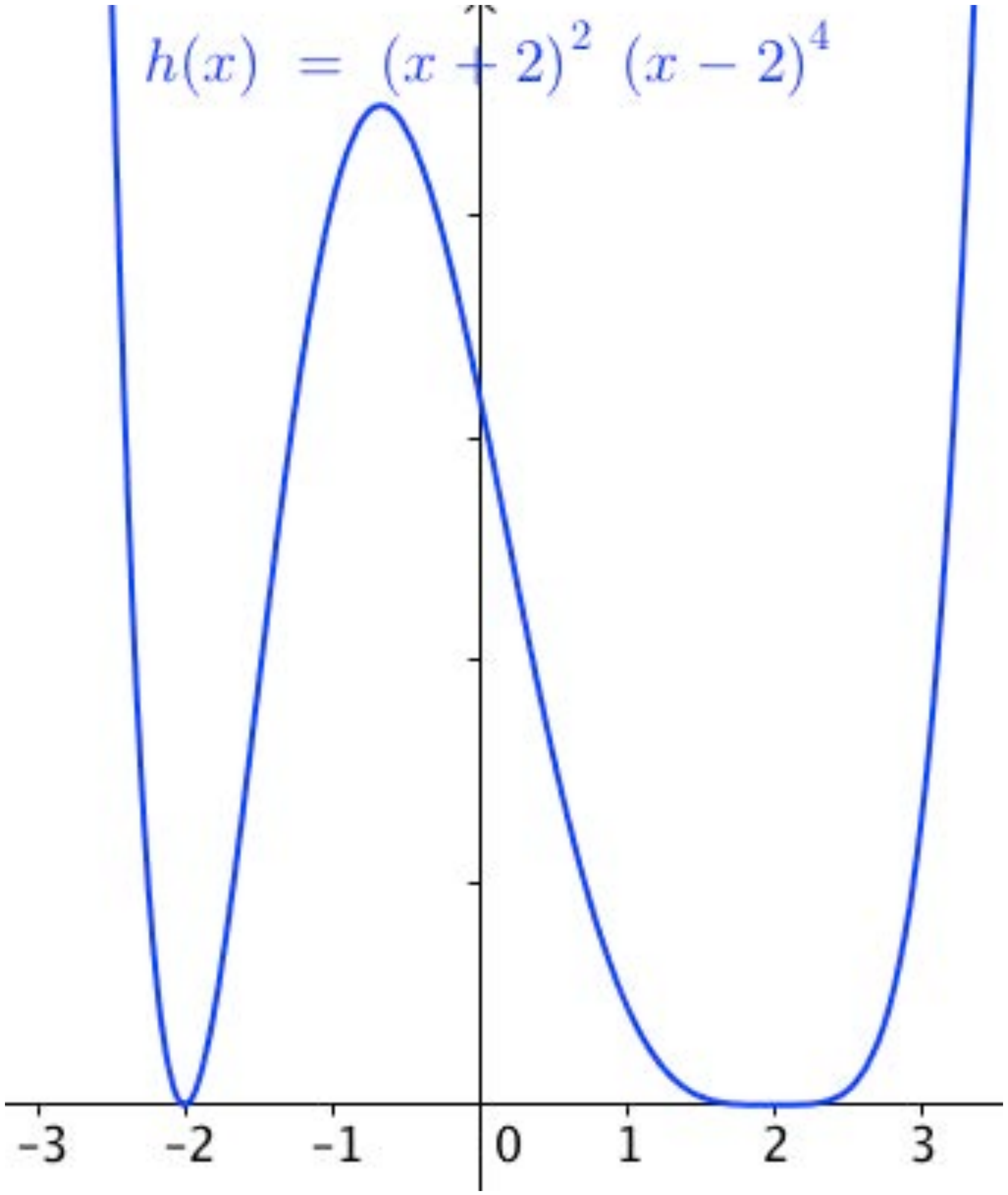
A vertex on the x-axis indicates a multiple zero, meaning the zero occurs two or more times.



This is a 4th-degree polynomial. It has two unique real zeros: -2 and 2. These two zeros are said to have a multiplicity of two, which means they each occur twice.

There are 4 real zeros and therefore no imaginary zeros for this function.

What do you think are the zeros and their multiplicity for this function?



Answer

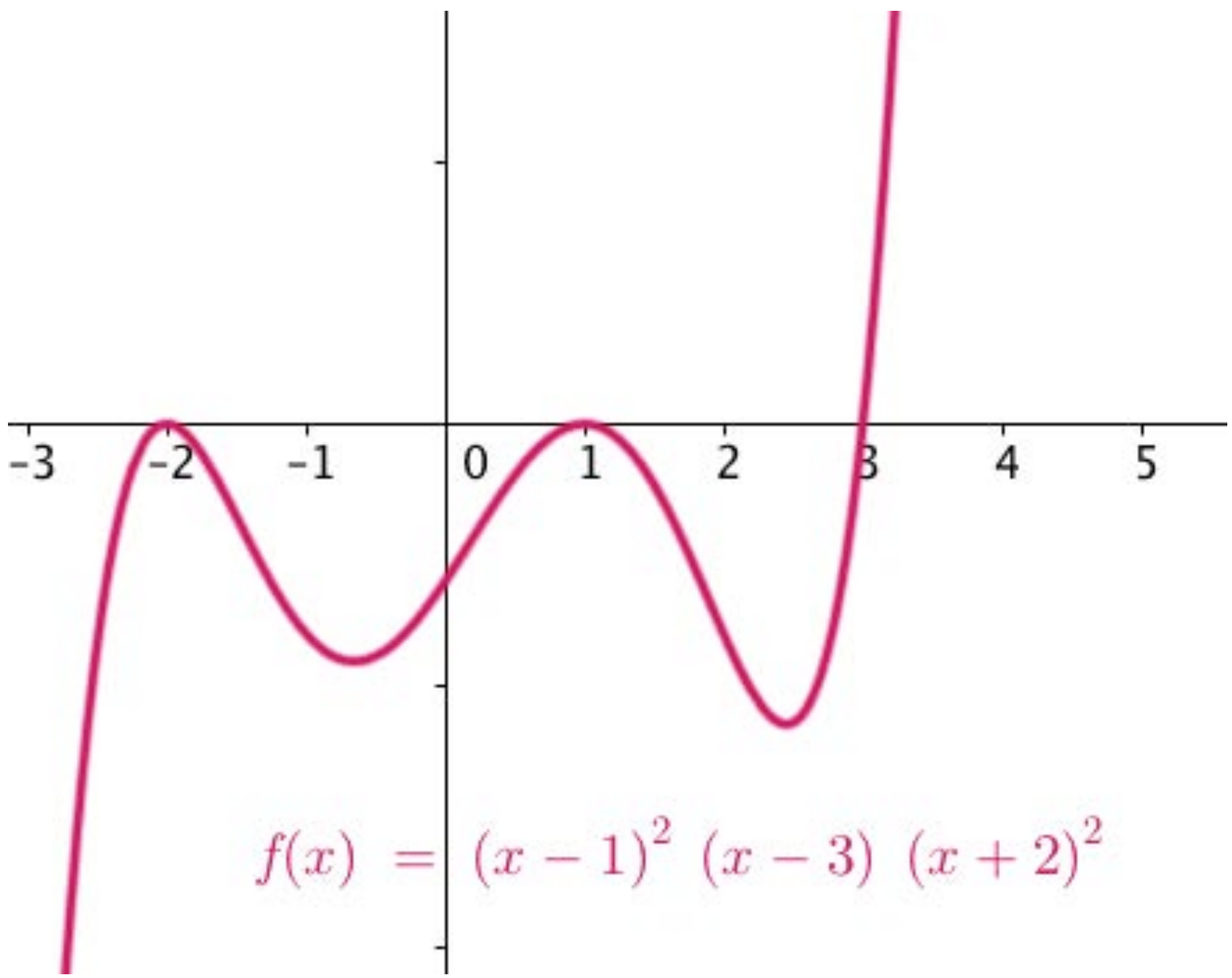
Notice the function for this graph.

x - 1 is a factor two times, and x = 1 is a zero twice.

x + 2 is a factor two times, and x = -2 is a zero twice.

Therefore, 1 and -2 are zeros with multiplicity of 2.

x + 3 is a factor once, and x = 3 is a zero with multiplicity of 1.



83 How many real zeros does the 4th-degree polynomial graphed have?

A 0

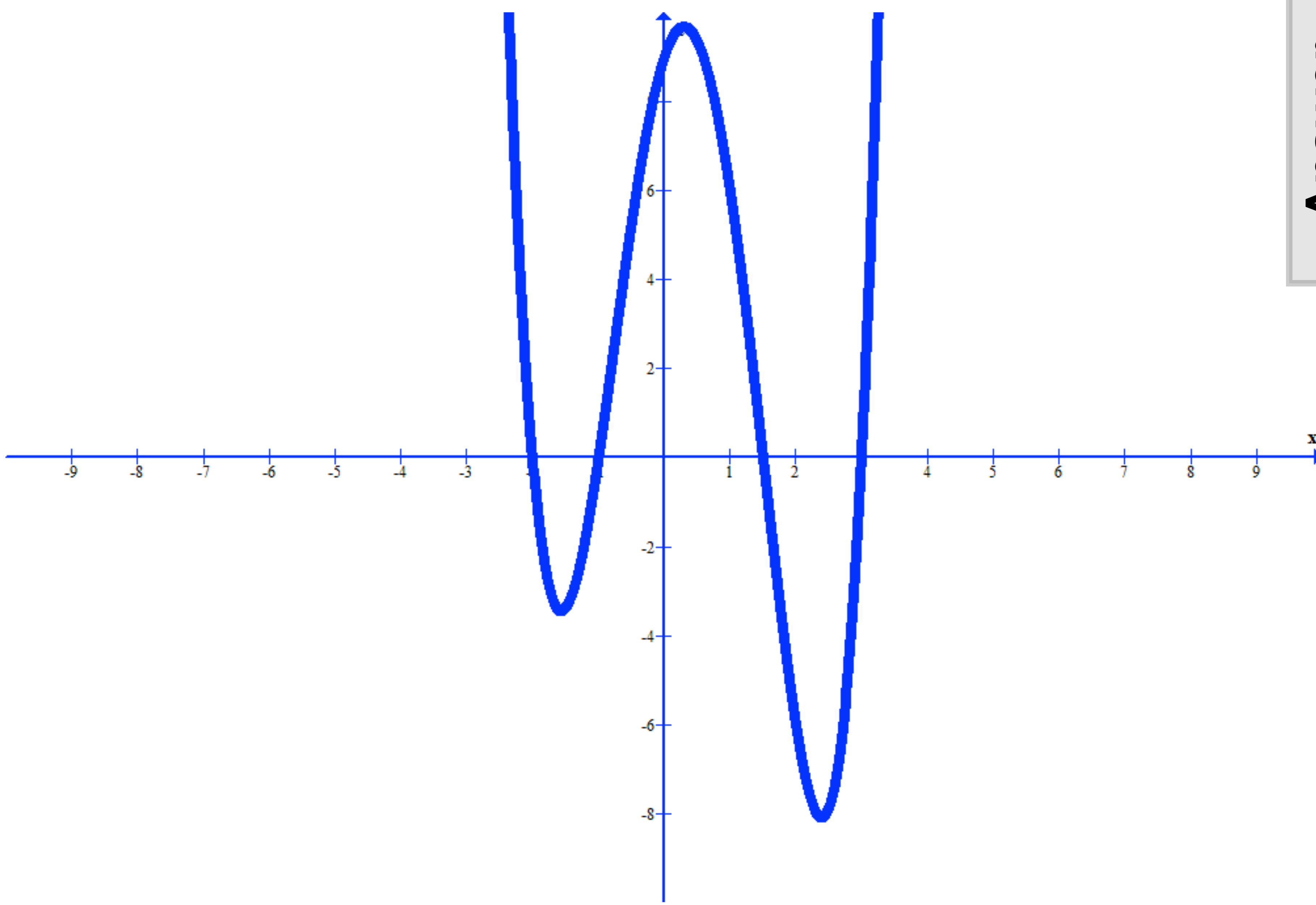
B 1

C 2

D 3

E 4

F 5

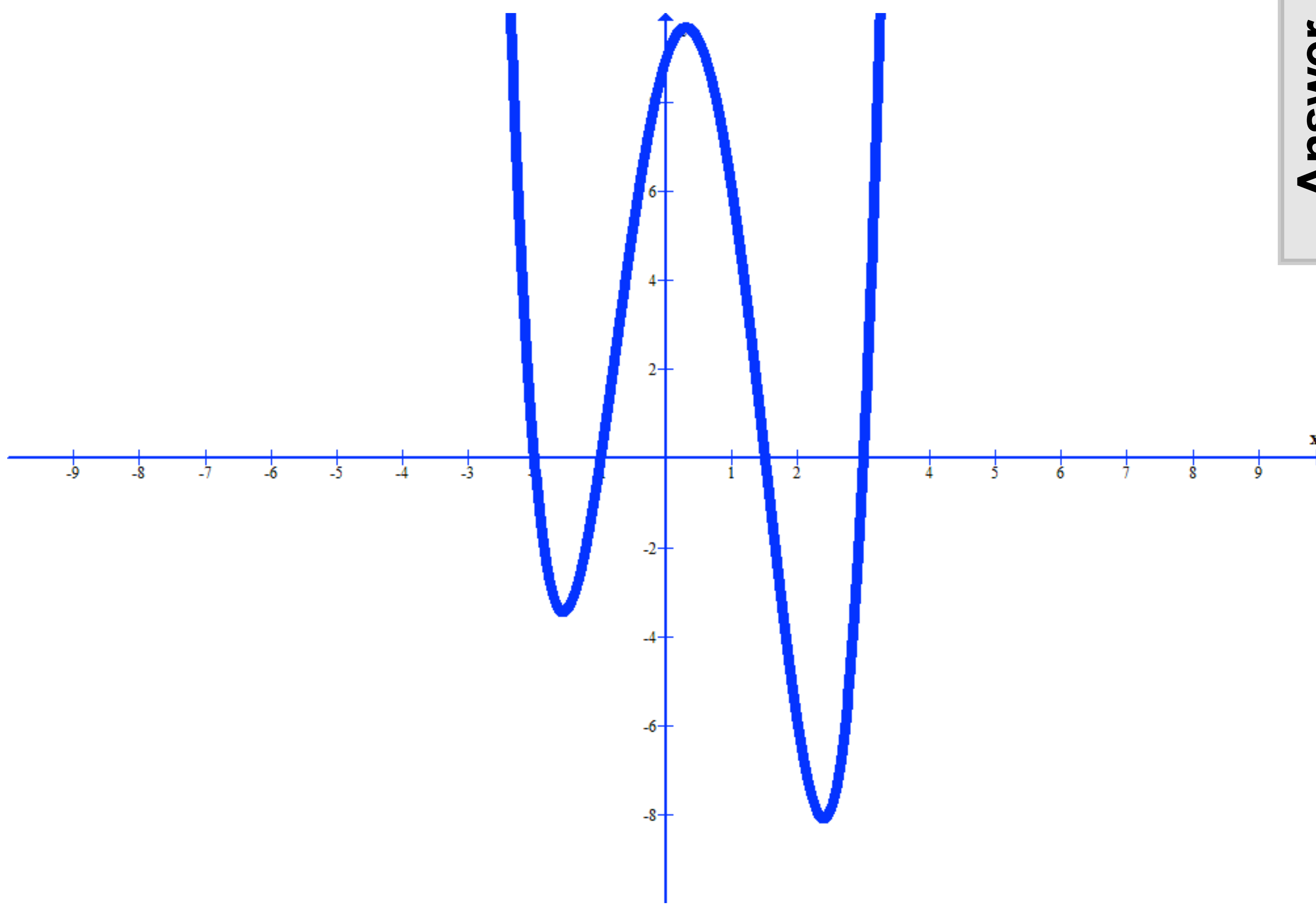


Answer

84 Do any of the zeros have a multiplicity of 2?

Yes

No



Answer

85 How many imaginary zeros does this 7th degree polynomial have?

A 0

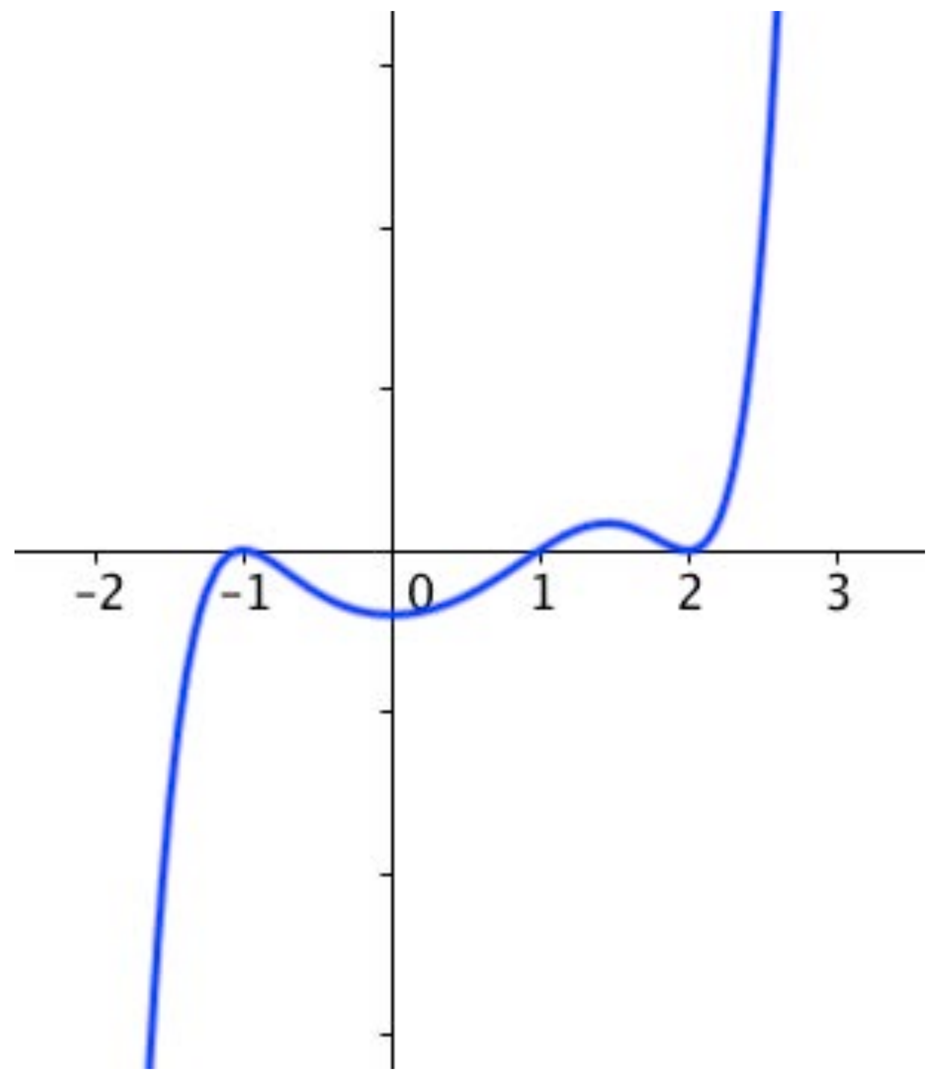
B 1

C 2

D 3

E 4

F 5



Answer

86 How many real zeros does the 3rd degree polynomial have?

A 0

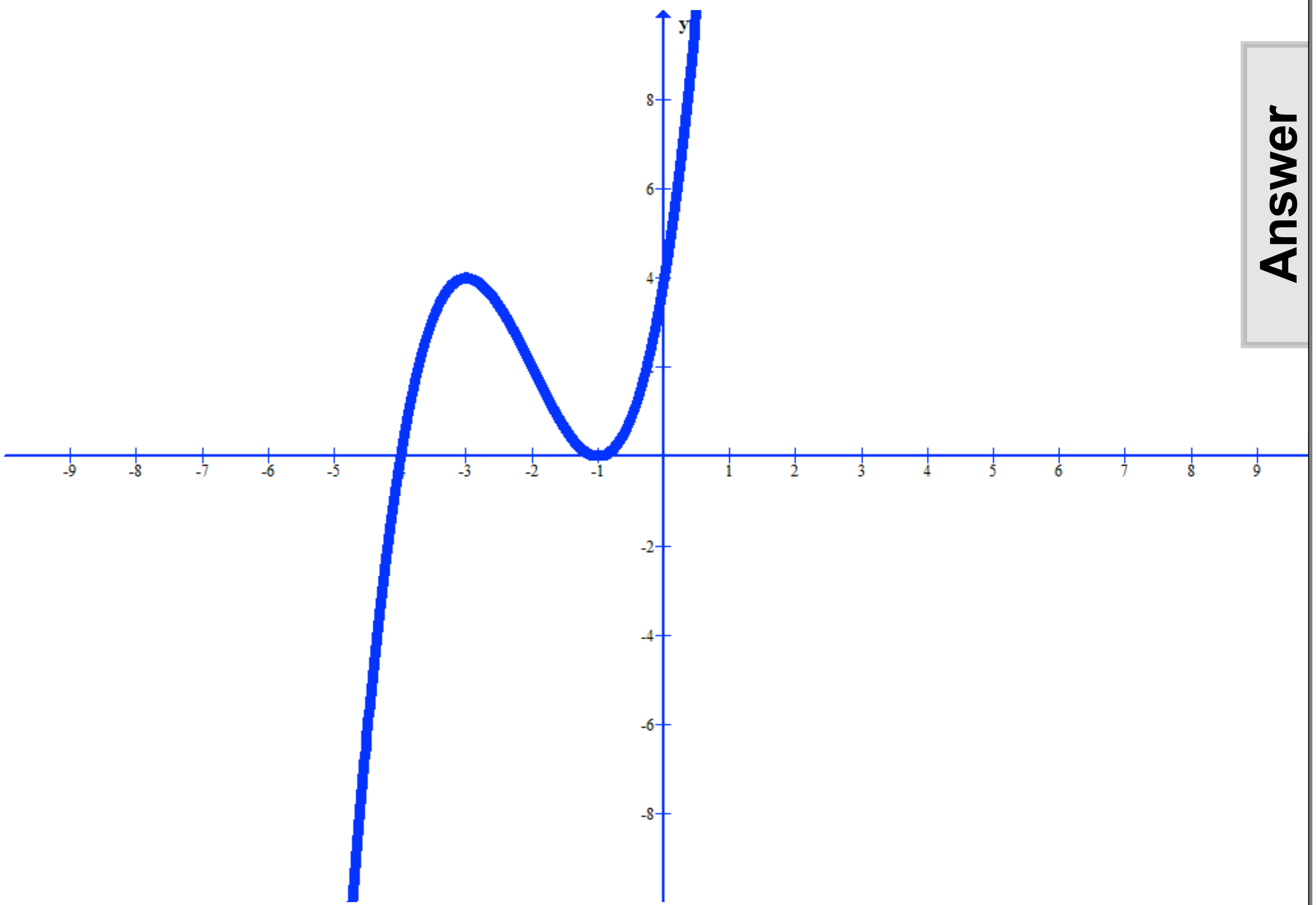
B 1

C 2

D 3

E 4

F 5

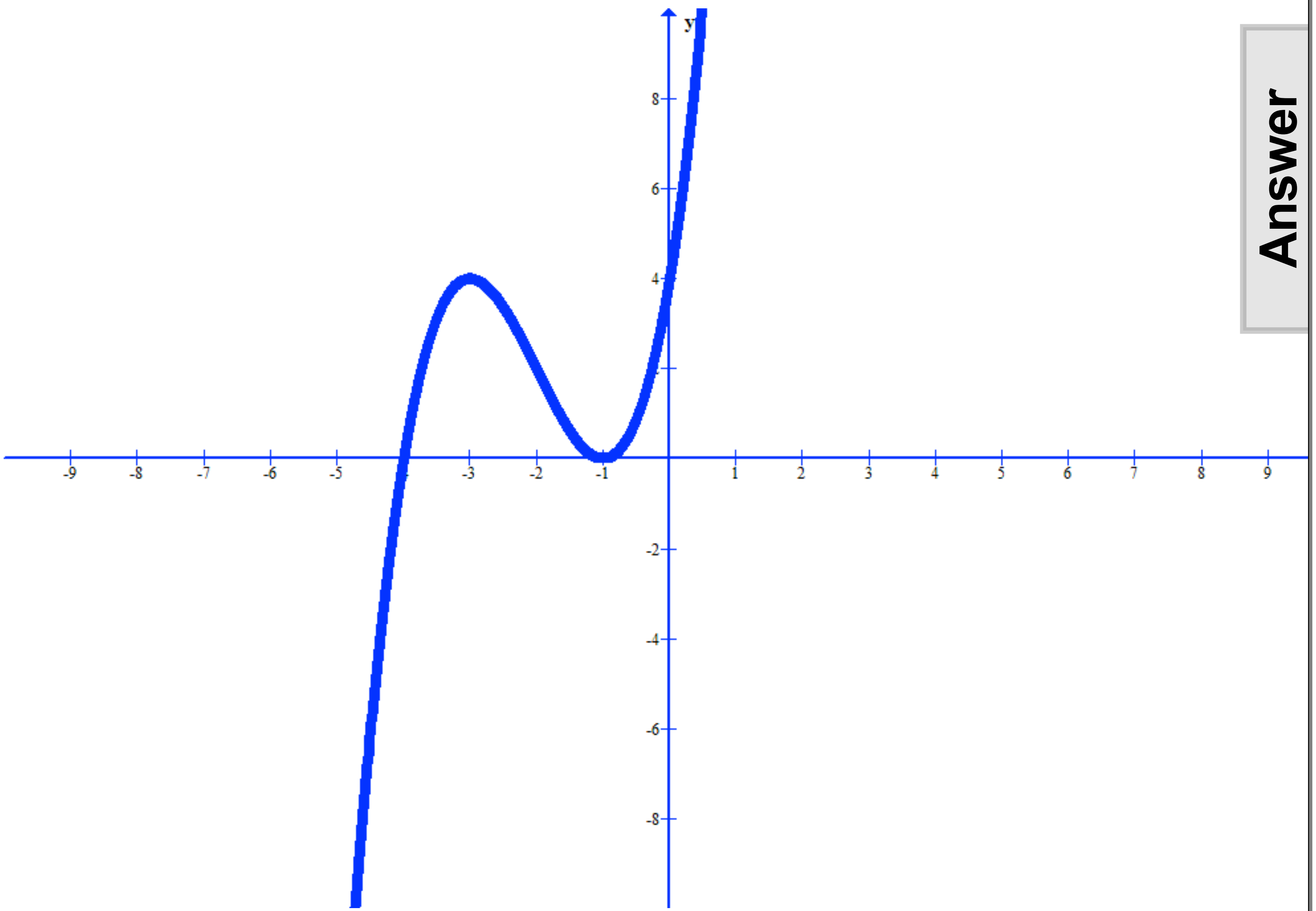


Answer

87 Do any of the zeros have a multiplicity of 2?

Yes

No



Answer

88 How many imaginary zeros does the 5th degree polynomial have?

A 0

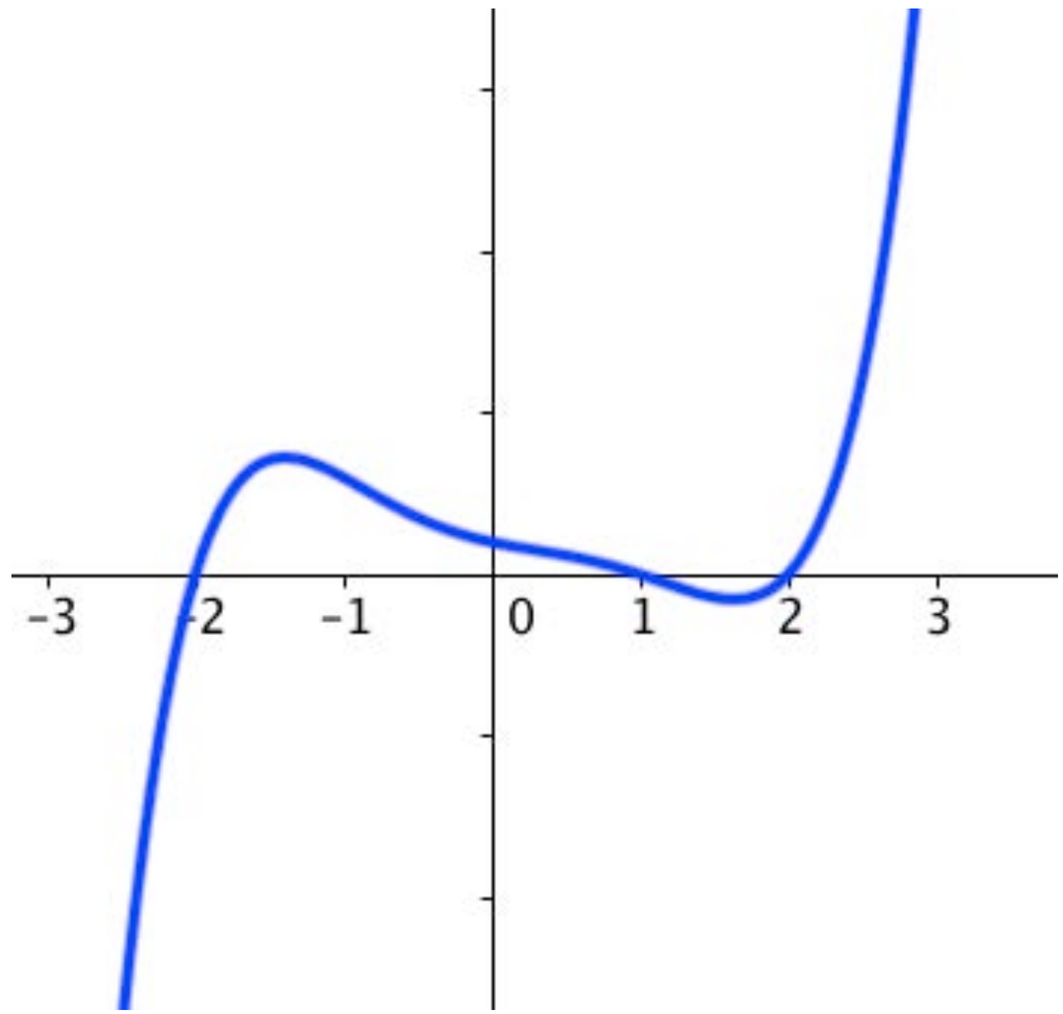
B 1

C 2

D 3

E 4

F 5



Answer

89 How many imaginary zeros does this 4th-degree polynomial have?

A 0

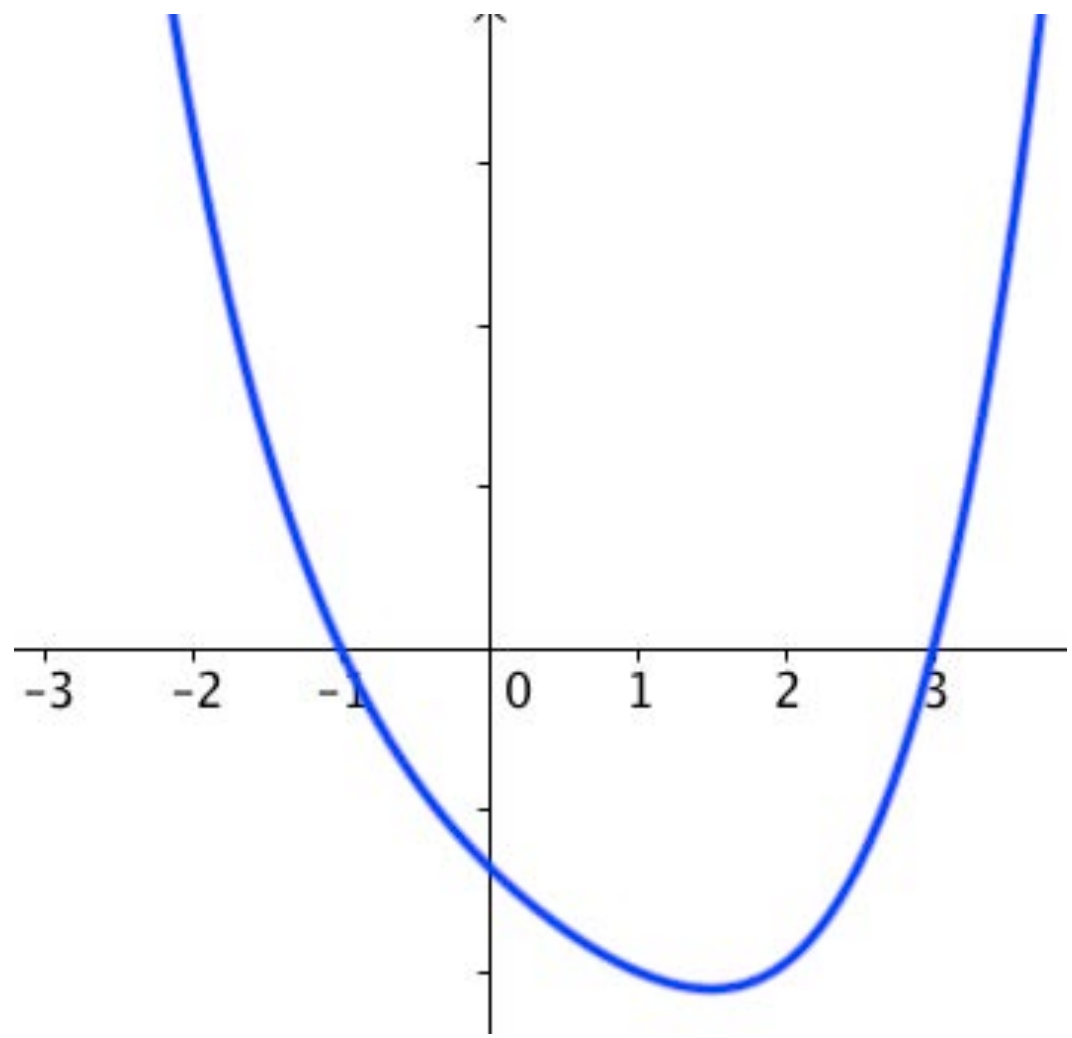
B 1

C 2

D 3

E 4

F 5



Answer

90 How many real zeros does the 6th degree polynomial have?

A 0

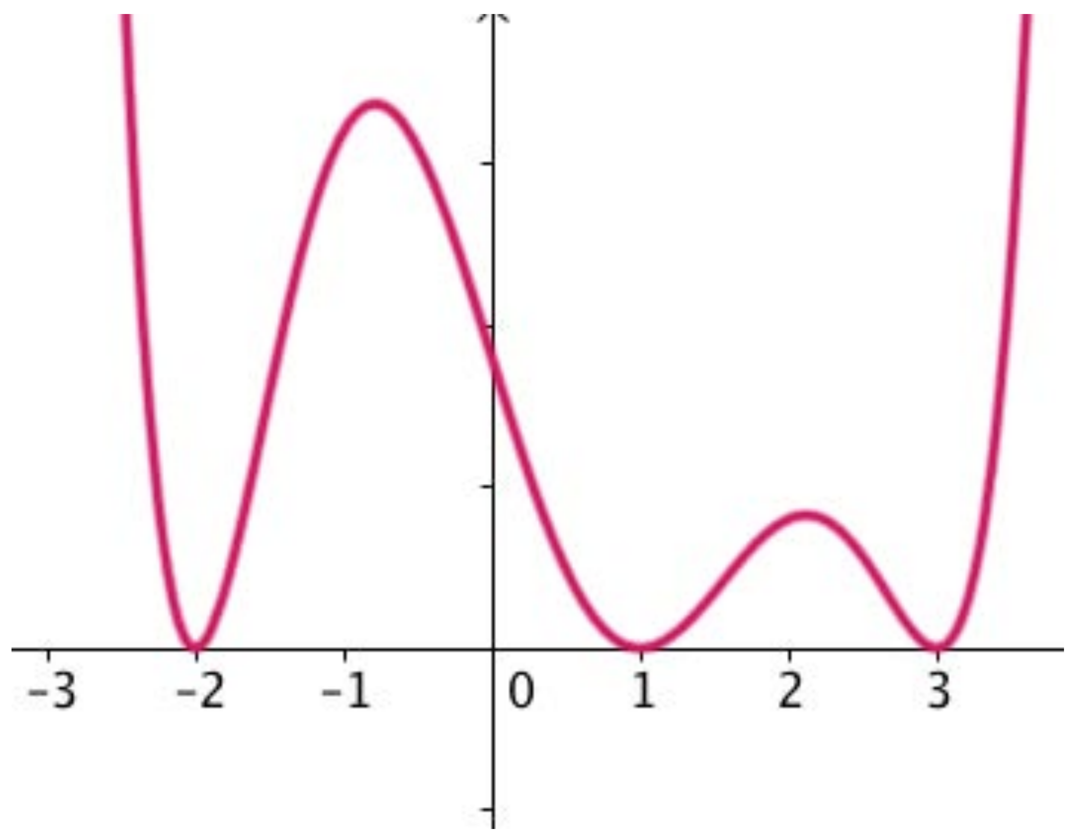
B 1

C 2

D 3

E 4

F 6

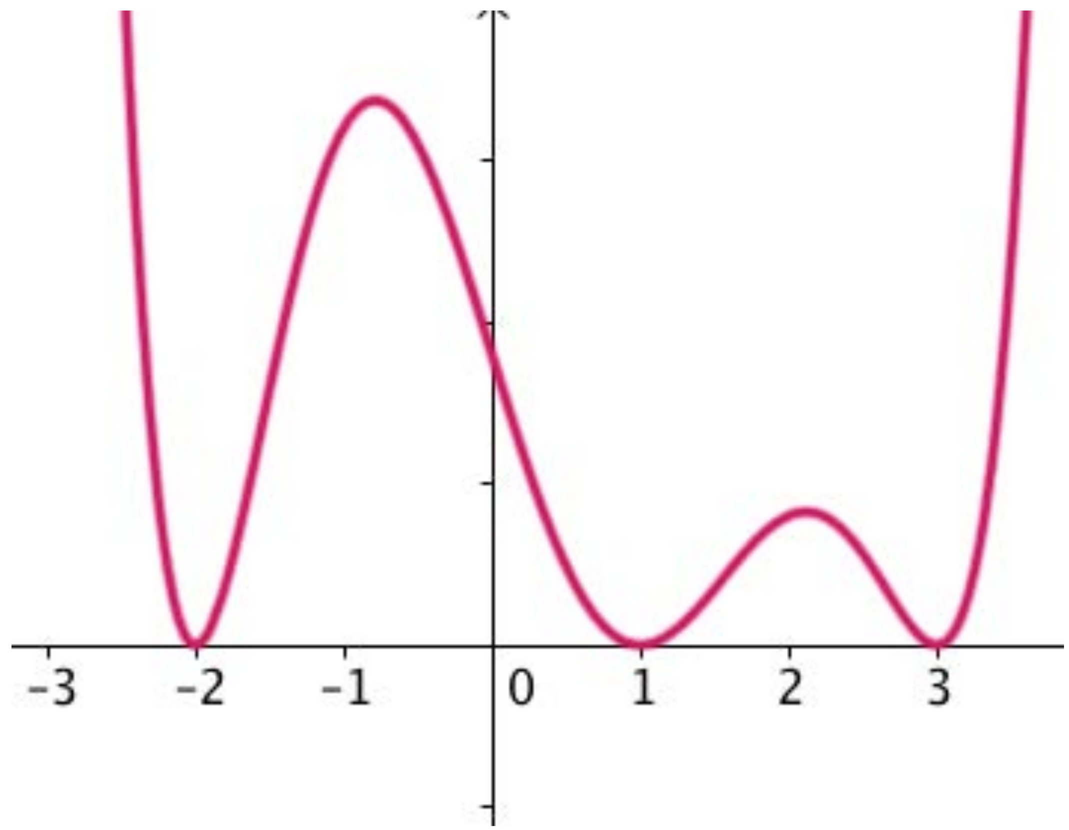


Answer

91 Do any of the zeros have a multiplicity of 2?

Yes

No



Answer

92 How many imaginary zeros does the 6th degree polynomial have?

A 0

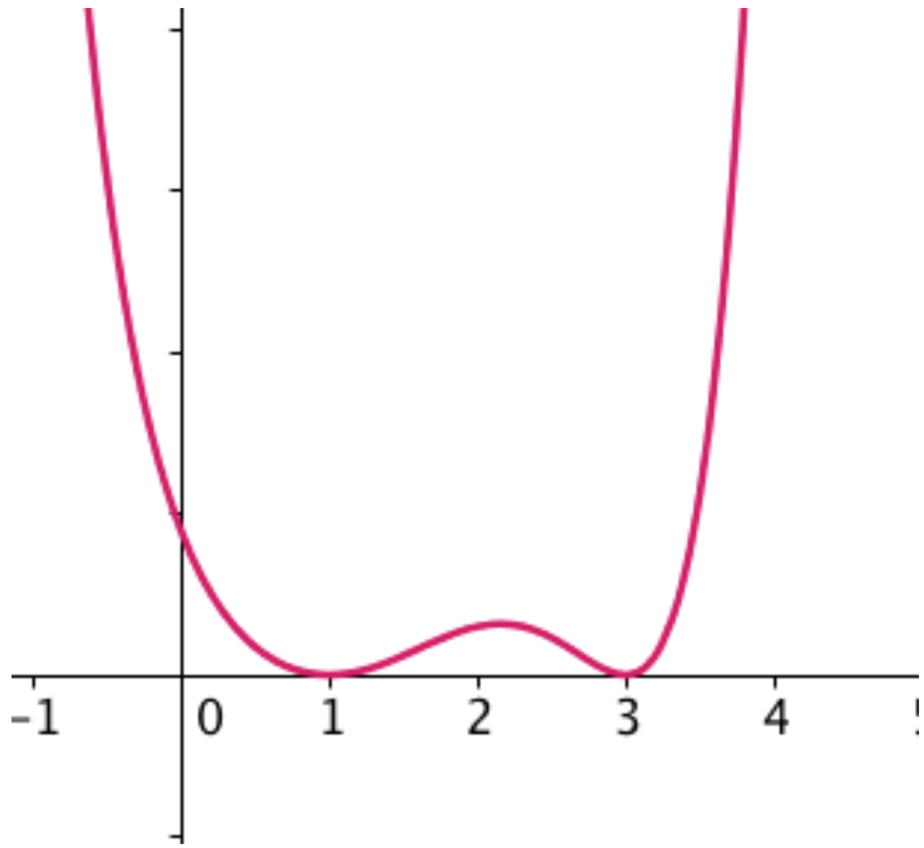
B 1

C 2

D 3

E 4

F 5



Answer

Finding the Zeros from an Equation in Factored Form:

Recall the Zero Product Property.

If the product of two or more quantities or factors equals 0, then at least one of the quantities must equal 0.

If $x(x+1)=0$, then $x=0$ or $x+1=0$.

If $(x+2)(x-3)=0$, then $x+2=0$ or $x-3=0$.

If $x(x+1)=0$, then $x=0$ or $x+1=0$.

So, if $f(x)=x(x+1)$, then the zeros of $f(x)$ are 0 and -1.

If $(x+2)(x-3)=0$, then $x+2=0$ or $x-3=0$.

So, if $f(x)=(x+2)(x-3)$, then the zeros of $f(x)$ are

____ and ____.

Find the zeros, including multiplicities, of the following polynomial.

What is the degree of the polynomial?

$$(x-5)(x^2-16)(x-2)^2(x+3)^2=0$$

$x+5=0$ or $x^2-16=0$ or $x-2=0$ or $x+3=0$

$x=-5$ $\quad$ $x^2=16$ $\quad$ $x=2$ $\quad$ $x=-3$

$x=\pm 4$

Answer

93 How many distinct real zeros does this polynomial have?

$$(x-2)(x-1)(x-3)(x-4)=0$$

A 0

B 1

C 2

D 3

E 4

F 5

Answer

94 What are the real zeros of

$$(x-2)(x-1)(x-3)(x-4)=0 \text{ ?}$$

A -4

B -3

C -2

D -1

E 0

F 1

G 2

H 3

I 4

Answer

Find the zeros, including multiplicities, of the following polynomial.

$$(x+6)(x^2-4)(x^2+9)(x-3)^2(x+4)^2=0$$

$x+6=0$ or $x^2-4=0$ or $x^2+9=0$ or $(x-3)^2=0$ or $(x+4)^2=0$

$x=-6$ $\quad x^2=4$ $\quad x^2=-9$ $\quad x-3=0$ $\quad x+4=0$

$x=\pm 2$ $\quad x=\pm 3i$ $\quad x=3$ $\quad x=-4$

Don't forget the ±!!

If $x^2=-9$, then $x=\pm\sqrt{-9}=\pm\sqrt{-1}\sqrt{9}=\pm 3i$.

Answer

Find the zeros, both real and imaginary, showing the multiplicities, of the following polynomial:

$$\left(x^2+1\right)\left(x-2\right)=0$$

click to reveal

95 What is the multiplicity of the root x = 1?

$$(x-2)(x-1)(x-3)(x-4)=0$$

Answer

96 How many distinct real zeros does the polynomial have?

$$(x-1)^2(x+3)(x+4)^2=0$$

A 0

B 1

C 2

D 3

E 4

F 5

Answer

97 How many distinct imaginary zeros does the polynomial have?

$$(x-1)^2(x+3)(x+4)^2=0$$

A 0

B 1

C 2

D 3

E 4

F 5

Answer

98 What is the multiplicity of x = 1?

$$(x-1)^2(x+3)(x+4)^2=0$$

Answer

99 What are the distinct real zeros of the polynomial?

$$(x^2 - 1)(x^2 + 4)(x + 7)^2 = 0$$

A -4

B 4

C 1

D -1

E -7

F 7

G 0

H 2

I -2

Answer

100 What are the imaginary roots of the polynomial?

$$(x^2-1)(x^2+4)(x+7)^2=0$$

A i

B -i

C 2i

D -2i

E 7i

F -7i

Answer

101 What is the multiplicity of x = 1?

$$(x^2 - 1)(x^2 + 4)(x + 7)^2 = 0$$

Answer

102 How many distinct real zeros does the polynomial have?

$$(x-2)^2(x^2+3)(x^2-4)(x+5)^2(x-1)^3=0$$

A 0

B 5

C 6

D 7

E 8

F 9

Answer

103 What is the multiplicity of x = 1?

$$(x-2)^2(x^2+3)(x^2-4)(x+5)^2(x-1)^3=0$$

Answer

104 How many distinct imaginary zeros does the polynomial have?

$$(x-2)^2(x^2+3)(x^2-4)(x+5)^2(x-1)^3=0$$

A 0

B 1

C 2

D 3

E 4

F 5

Answer

Find the zeros, showing the multiplicities, of the following polynomial.

$$x^3 - 2x^2 + x = 0$$

To find the zeros, you must first write the polynomial in factored form.

$$x^3 - 2x^2 + x = 0$$
$$x(x^2 - 2x + 1) = 0$$
$$x(x - 1)(x - 1) = 0$$

$x = 0$ or $(x - 1) = 0$ or $(x - 1) = 0$

$\boxed{x = 0}$ or $\boxed{x = 1}$ or $\boxed{x = 1}$

This polynomial has two distinct real zeros: 0 and 1.

This is a 3rd degree polynomial, so there are 3 zeros (count 1 twice).
1 has a multiplicity of 2.
0 has a multiplicity of 1.
There are no imaginary zeros.

Find the zeros, including multiplicities, of the following polynomial.

$$x^4 - 6x^2 - 27 = 0$$

$$(x^2 - 9)(x^2 + 3) = 0$$

$(x^2 - 9) = 0$ or $(x^2 + 3) = 0$

$x^2 = 9$ or $x^2 = -3$

$\boxed{x = \pm 3}$ or $\boxed{x = \pm i\sqrt{3}}$

This polynomial has 4 zeros.
There are two distinct real zeros: ± 3, both with a multiplicity of 1.
There are two imaginary zeros: $\pm i\sqrt{3}$, both with a multiplicity of 1.

105 How many zeros does the polynomial function $f(x) = x^3 - x^2 - 6x$ have?

A 0

B 1

C 2

D 3

E 4

Answer

106 How many REAL zeros does the polynomial equation $x^3 - x^2 - 6x = 0$ have?

A 0

B 1

C 2

D 3

E 4

Answer

107 What are the zeros and their multiplicities of the polynomial function $f(x) = x^3 - x^2 - 6x$?

A x = -2, mulitplicity of 1

B x = -2, multiplicity of 2

C x = 3, multiplicity of 1

D x = 3, multiplicity of 2

E x = 0, multiplicity of 1

F x = 0, multiplicity of 2

Answer

108 Find the solutions of the following polynomial equation, including multiplicities. $3x^3 - 18x^2 + 27x = 0$

A x = 0, multiplicity of 1

B x = 3, multiplicity of 1

C x = 0, multiplicity of 2

D x = 3, multiplicity of 2

Answer

109 Find the zeros of the polynomial equation, including multiplicities: $x^3 - 2x^2 + x - 2 = 0.$

A x = 2, multiplicity 1

B x = 2, multiplicity 2

C x = -i, multiplicity 1

D x = i, multiplicity 1

E x = -i, multiplplcity 2

F x = i, multiplicity 2

Answer

110 Find the zeros of the polynomial equation, including multiplicities:

$$x^3 + 2x^2 - 2x - 4 = 0$$

A 2, multiplicity of 1

B 2, multiplicity of 2

C -2, multiplicity of 1

D -2, multiplicity of 2

E $\sqrt{2}$, multiplicity of 1

F $-\sqrt{2}$, multiplicity of 1

Answer

Find the zeros, showing the multiplicities, of the following polynomial.

$$3x^3 + 4x^2 - 5x - 2 = 0$$

To find the zeros, you must first write the polynomial in factored form.

However, this polynomial cannot be factored using normal methods.

What do you do when you are STUCK??

We are going to need to do some long division, but by what do we divide?

The Remainder Theorem told us that for a function, $f(x)$, if we divide $f(x)$ by $x - a$, then the remainder is $f(a)$. If the remainder is 0, then $x - a$ if a factor of $f(x)$.

In other words, if $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

So how do we figure out what a should be????

We could use guess and check, but how can we narrow down the choices?