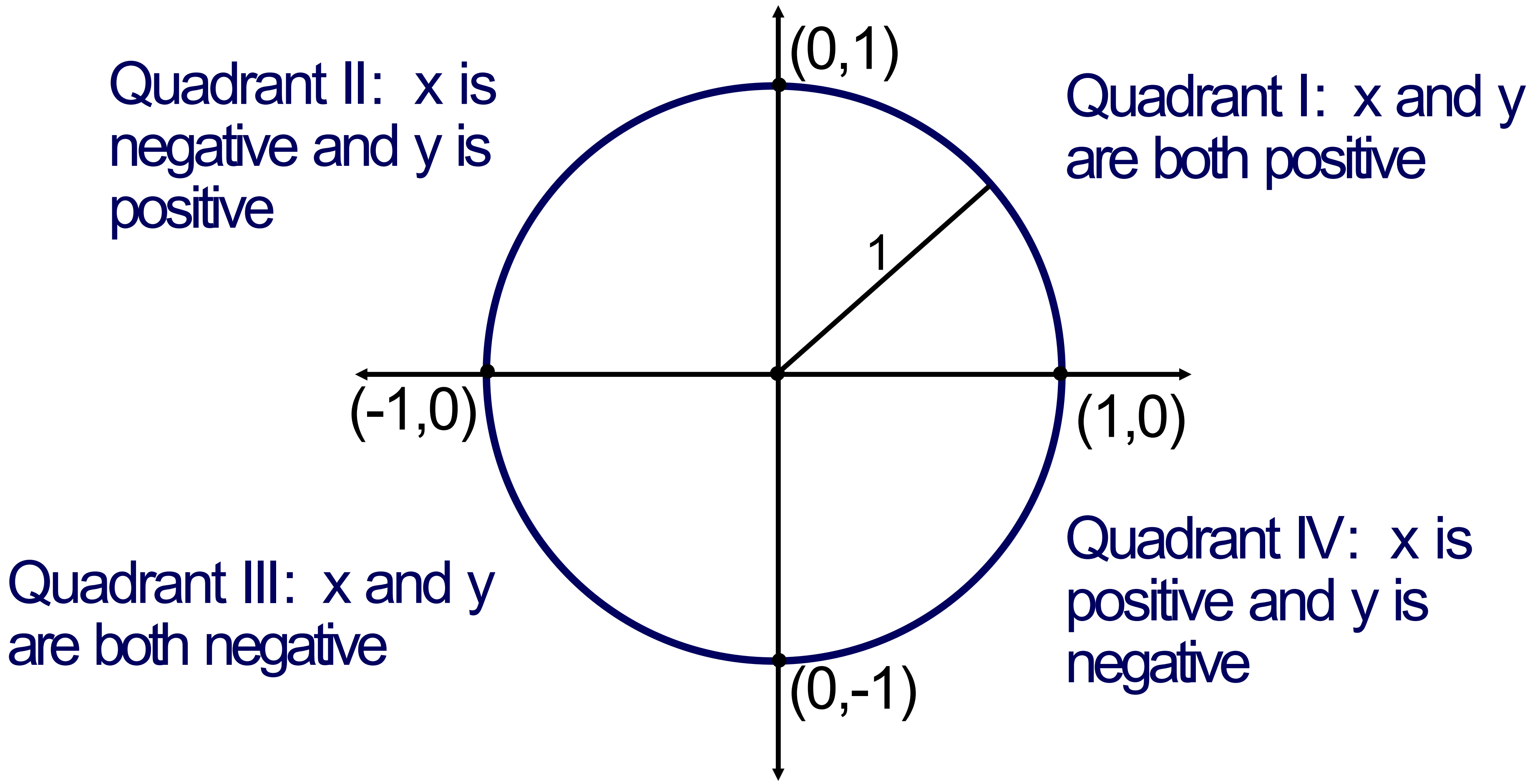
Unit Circle

Return to Table of Contents

The Unit Circle

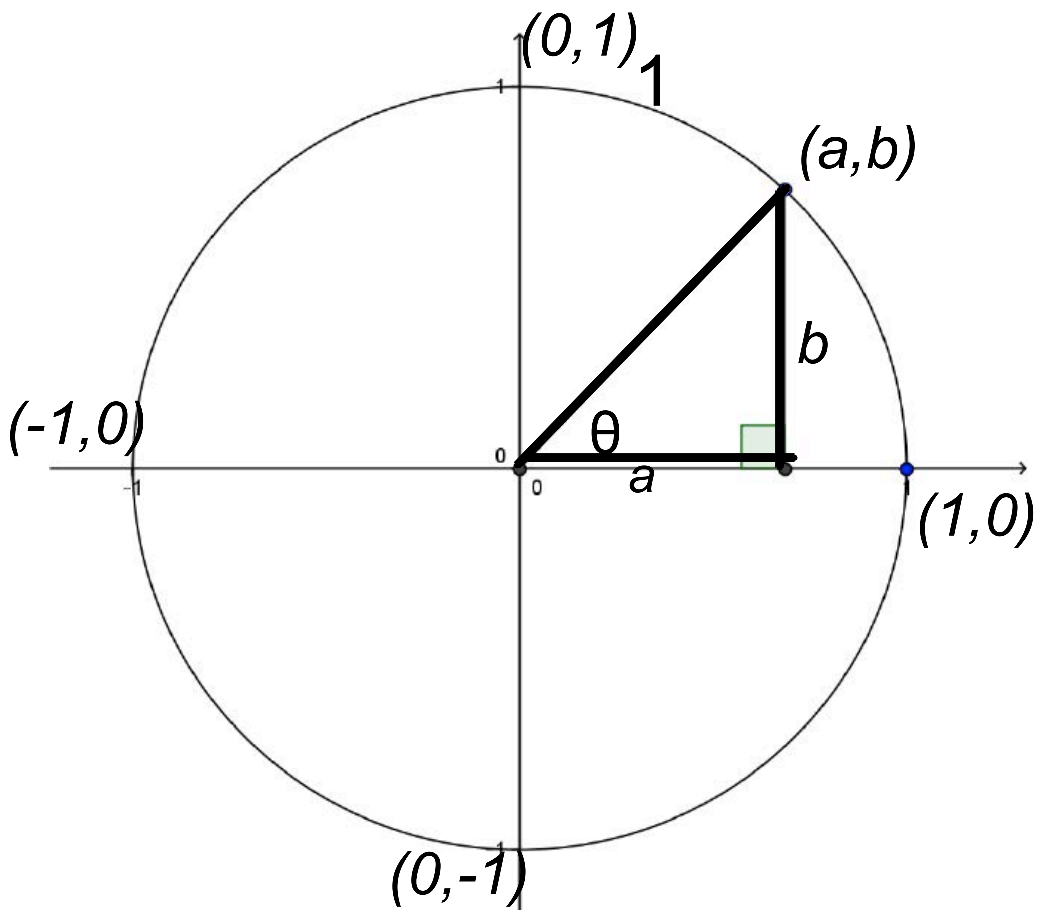
The circle $x^2 + y^2 = 1$, with center (0,0) and radius 1,

is called the **unit circle**.



The Unit Circle

The unit circle allows us to extend trigonometry beyond angles of triangles to angles of all measures.



In this triangle,

$\sin\theta = \dfrac{b}{1} = b$

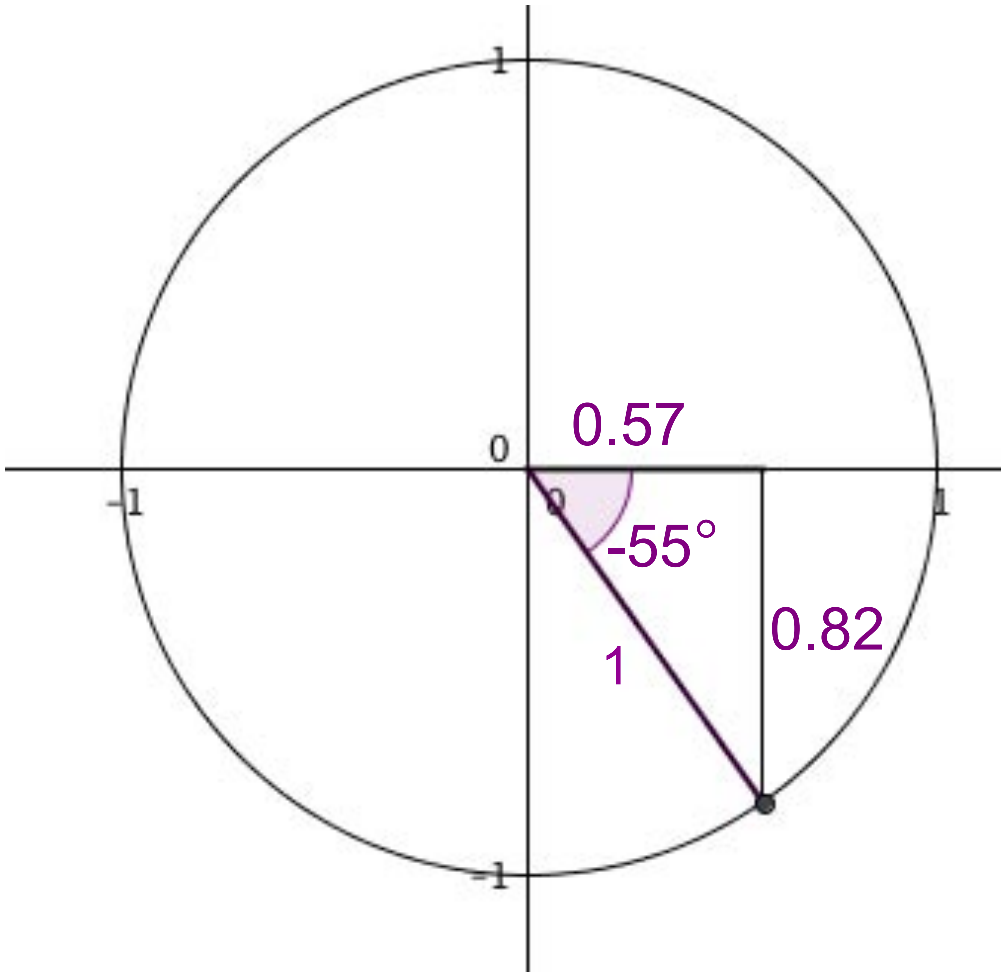
$\cos\theta = \dfrac{a}{1} = a$

so the coordinates of (a,b) are also (cosθ, sinθ)

For any angle in standard position, the point where the terminal side of the angle intercepts the circle is called the **terminal point**.

Terminal Point

In this example, the terminal point is in Quadrant IV.



If we look at the triangle, we can see that
$\sin(-55^\circ) = 0.82$
$\cos(-55^\circ) = 0.57$
EXCEPT that we have to take the direction into account, and so $\sin(-55^\circ)$ is negative because the y value is below the x-axis.

For any angle θ in standard position, the terminal point has coordinates $(\cos\theta, \sin\theta)$.

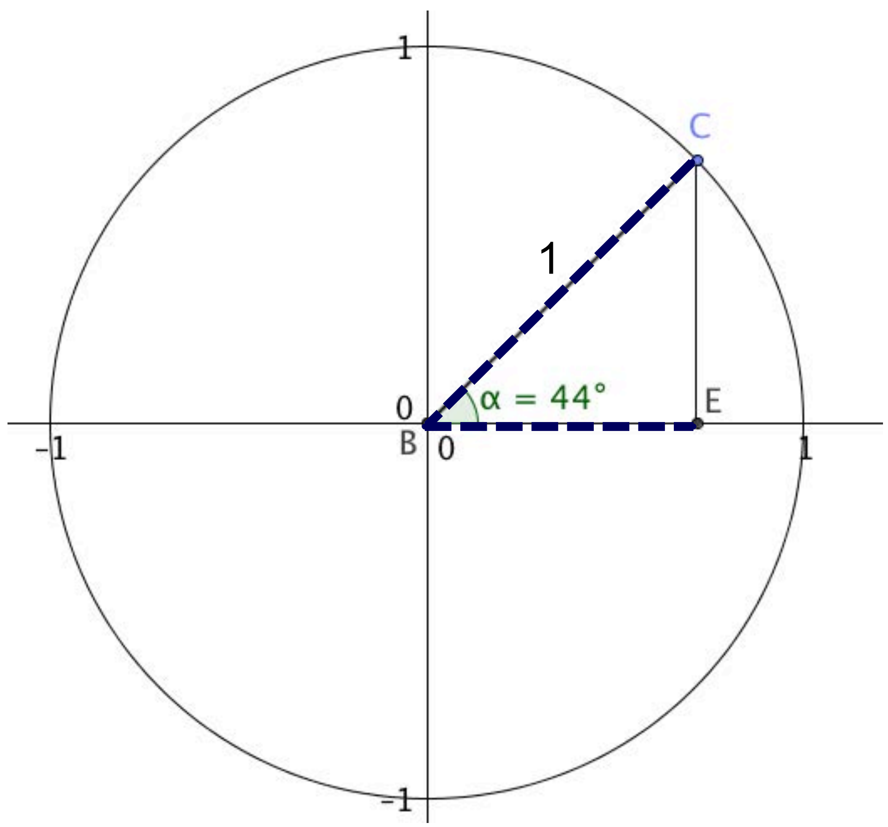
Unit Circle

Click on this text to go to the Khan Academy Unit Circle Manipulative try some problems:

Finding Coordinates of a Point

What are the coordinates of point C?

In this example, we know the angle. Using a calculator, we find that cos 44° ≈ 0.72 and sin 44° ≈ 0.69, so the coordinates of C are approximately (0.72, 0.69).



Note that $0.72^2 + 0.69^2 \approx 1$!

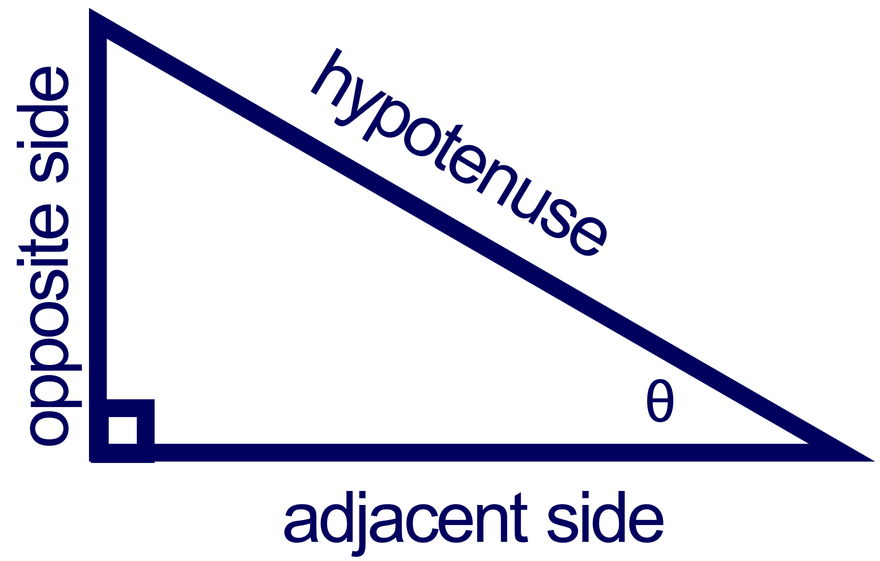
The Tangent Function

Recall SOH-CAH-TOA

$$\sin\theta = \frac{opp}{hyp}$$

$$\cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{opp}{adj}$$



It is also true that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Why?

$$\frac{\frac{opp}{hyp}}{\frac{adj}{hyp}} = \frac{opp}{\cancel{hyp}} \cdot \frac{\cancel{hyp}}{adj} = \frac{opp}{adj} = \tan\theta$$

Angles in the Unit Circle

Example: Given a terminal point on the unit circle $\left(-\frac{9}{41}, \frac{40}{41}\right)$.

Find the value of cos, sin and tan of the angle.

Solution: Let the angle be θ.

x = cos θ , so cos $\theta = -\frac{9}{41}$

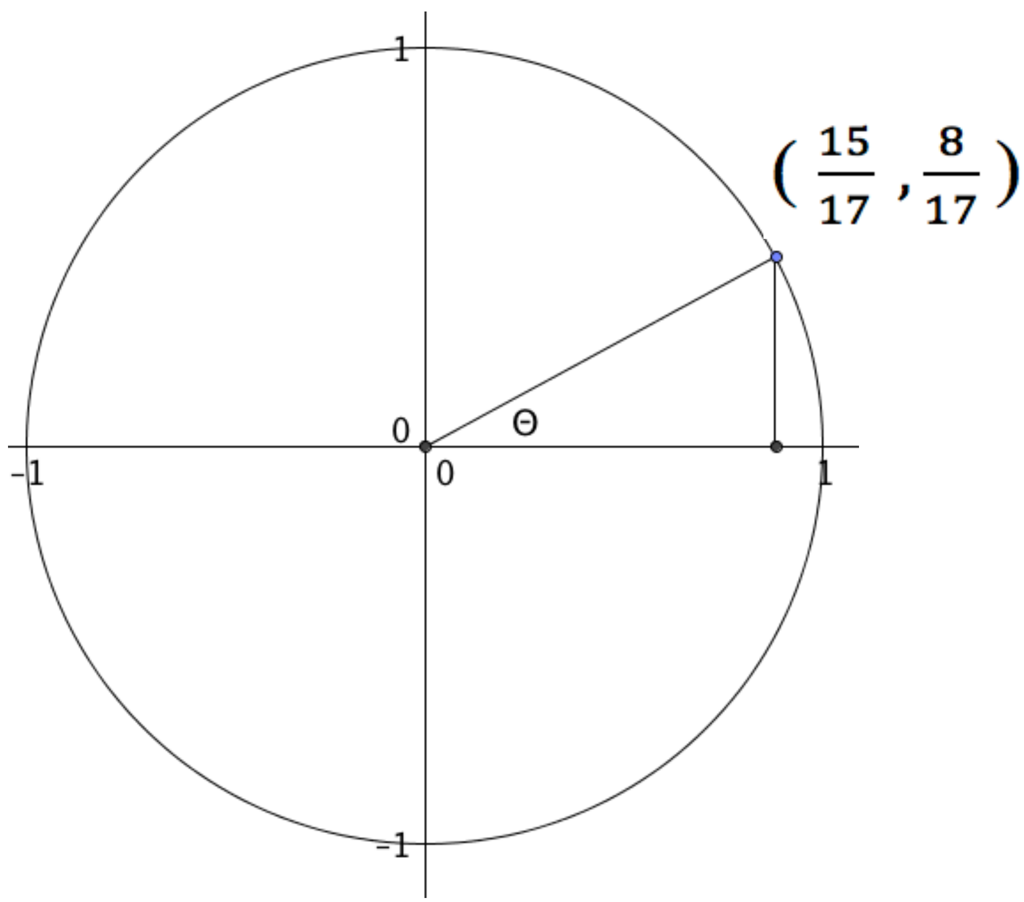
y = sin θ, so sin $\theta = \frac{40}{41}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{40}{41}}{-\frac{9}{41}} = \frac{40}{\cancel{41}} \cdot -\frac{\cancel{41}}{9} = -\frac{40}{9}$$

(Shortcut: Just cross out the 41's in the complex fraction.)

Finding Cos, Sin, and Tan

Example: Given a terminal point $\left(\frac{15}{17}, \frac{8}{17}\right)$ find θ, $\tan\theta$ and $\csc\theta$.



To find θ, use $\sin^{-1}$ or $\cos^{-1}$:

$$\sin^{-1}\left(\frac{8}{17}\right) = \theta$$

$$\theta \approx 28.1^\circ$$

$$\tan\theta = \sin\theta / \cos\theta$$

$$\tan\theta = \frac{8}{15}$$

$$\csc\theta = 1/\sin\theta$$

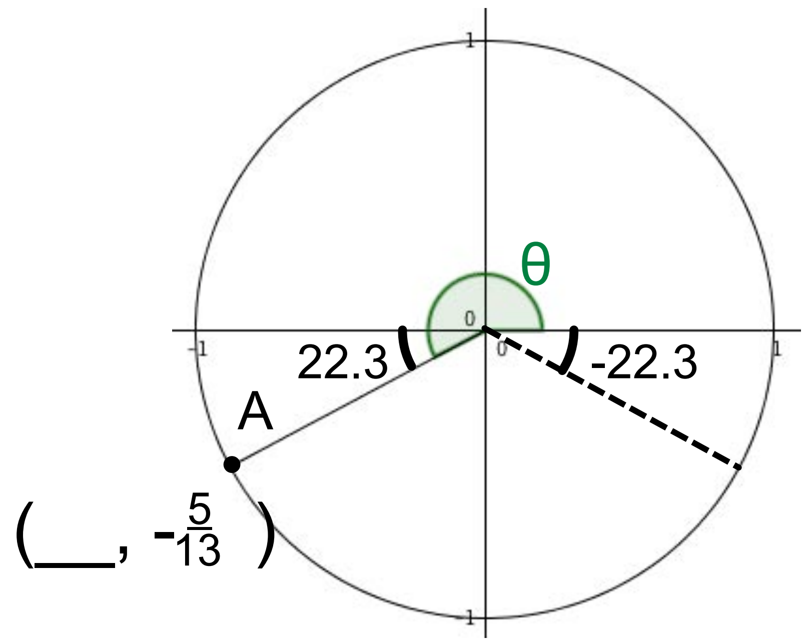
$$\csc\theta = \frac{17}{8}$$

Note the "hidden" Pythagorean Triple, 8, 15, 17).

Finding Cos, Sin, and Tan

Example: Find the x-value of point A, θ and the tan θ.

For every point on the circle,



$$x^2 + y^2 = 1$$

$$x^2 + \left(-\frac{5}{13}\right)^2 = 1$$

$$x^2 + \frac{25}{169} = 1$$

$$x^2 = \frac{169}{169} - \frac{25}{169} = \frac{144}{169}$$

$$x = \pm\frac{12}{13}$$

Since x is in quadrant III, x = $-\frac{12}{13}$

$\sin^{-1}\left(-\frac{5}{13}\right) \approx -22.3°$, BUT θ is in quadrant III, so

θ = 180 + 22.3 = 202.3° (notice how 202.3° and -22.3° have the same sine)

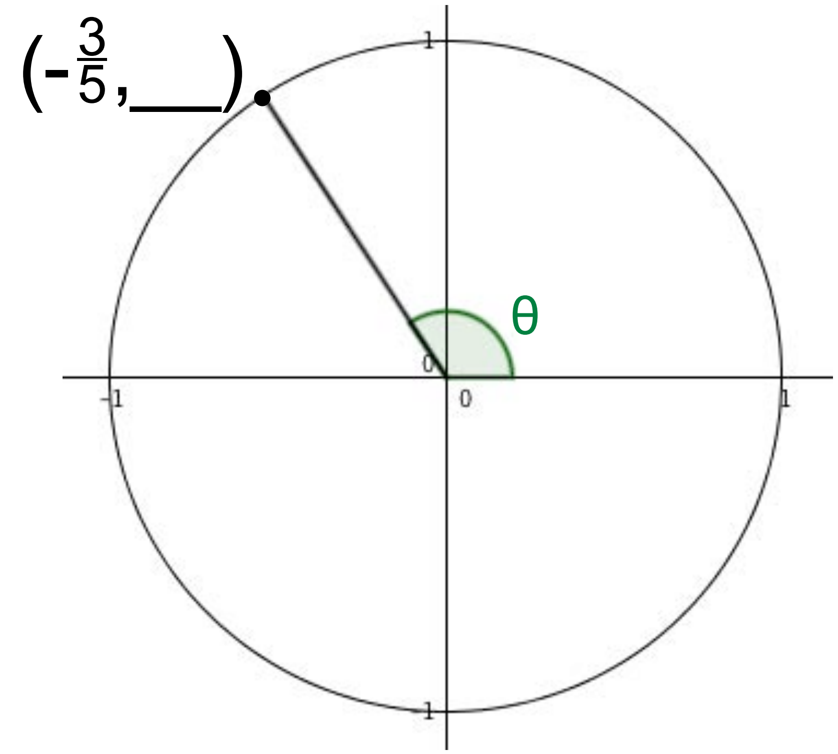
$$\tan\ \theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{5}{13}}{-\frac{12}{13}} = \frac{5}{12}$$

Finding Cos, Sin, and Tan

Example: Given the terminal point of $(^{-5}/_{13}, ^{-12}/_{13})$. Find sin x, cos x, and tan x.

Answer

14 What is tan θ?



A $\frac{3}{4}$

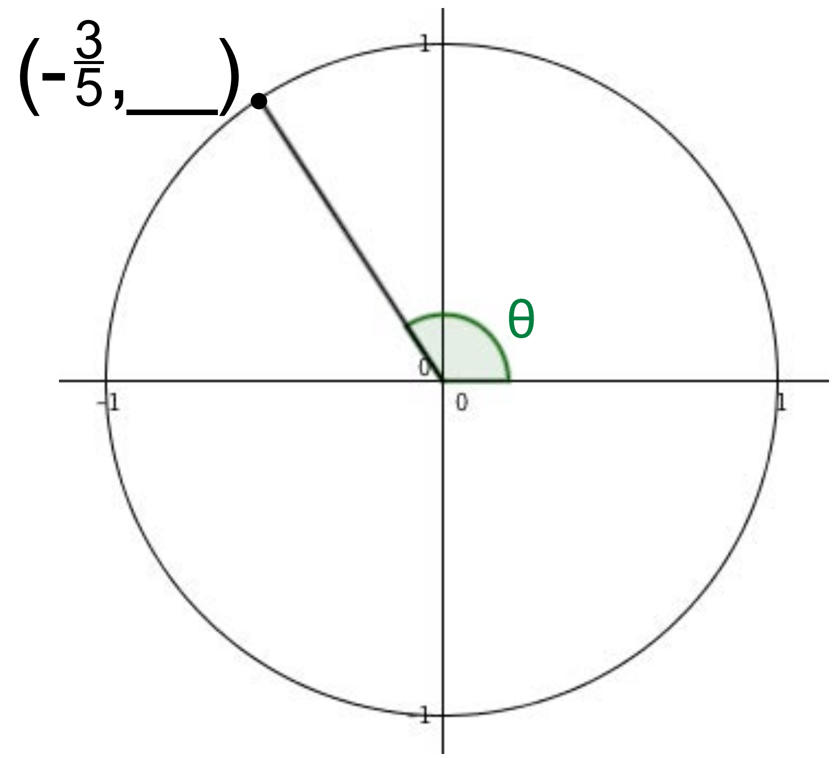
B $\frac{4}{3}$

C $-\frac{3}{4}$

D $-\frac{4}{3}$

Answer

15 What is sin θ?



A $\frac{3}{4}$

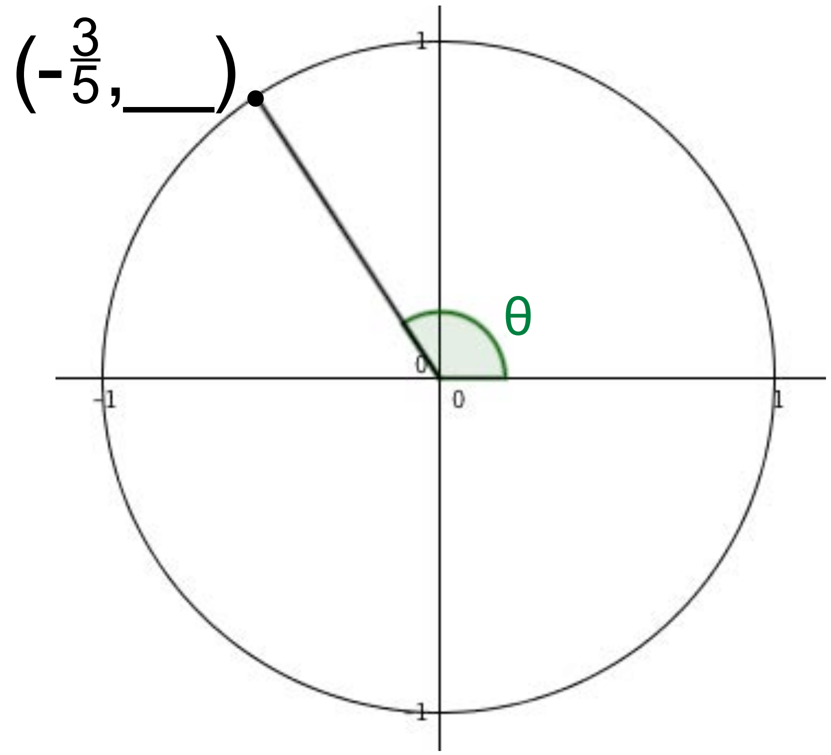
B $\frac{4}{5}$

C $-\frac{3}{4}$

D $-\frac{4}{5}$

Answer

16 What is θ (give your answer to the nearest degree)?



Answer

17 Given the terminal point $\left(-\frac{7}{25}, -\frac{24}{25}\right)$, find tan x.

Answer

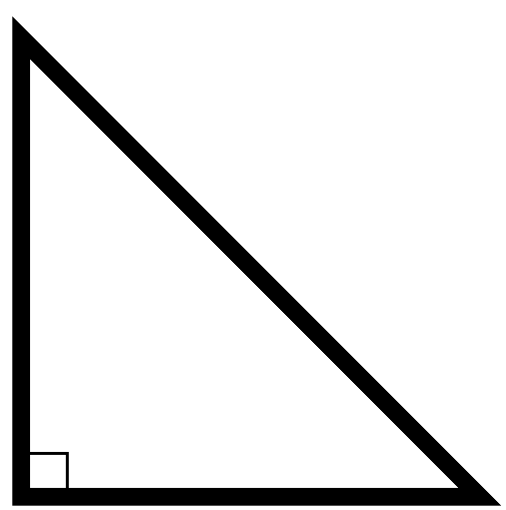
18 Knowing tan x = $-\frac{9}{40}$

Find sin x if the terminal point is in the 2^{nd} quadrant

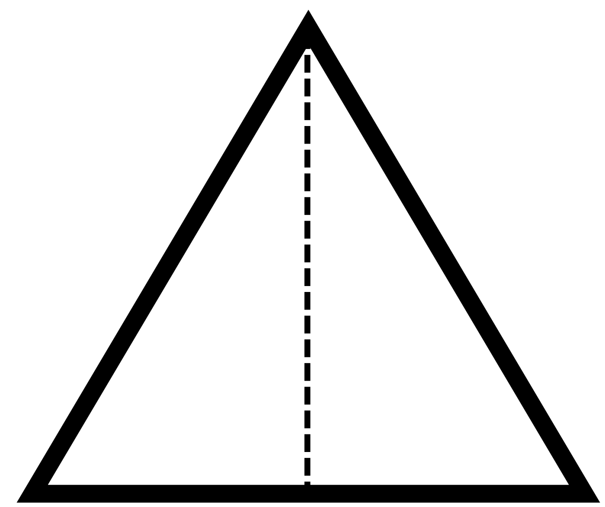
Answer

Equilateral and Isosceles Triangles

Equilateral and isosceles triangles occur frequently in geometry and trigonometry. The angles in these triangles are multiples of 30° and 45°. A calculator will give approximate values for the trig functions of these angles, but we often need to know the exact values.



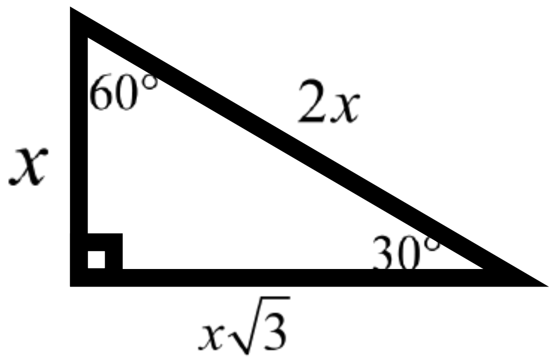
Isosceles Right Triangle

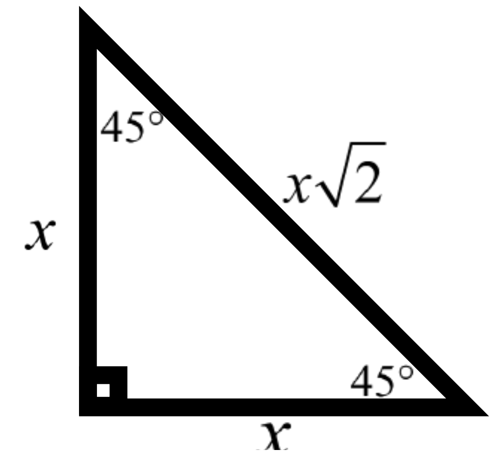


Equilateral Triangle

(the altitude divides the triangle into two 30-60-90 triangles)

Special Right Triangles





$$\cos(30^\circ) = \frac{\sqrt{3}}{2} \qquad \cos(60^\circ) = \frac{1}{2}$$

$$\sin(30^\circ) = \frac{1}{2} \qquad \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

$$\tan(30^\circ) = \frac{\sqrt{3}}{3} \qquad \tan(60^\circ) = \sqrt{3}$$

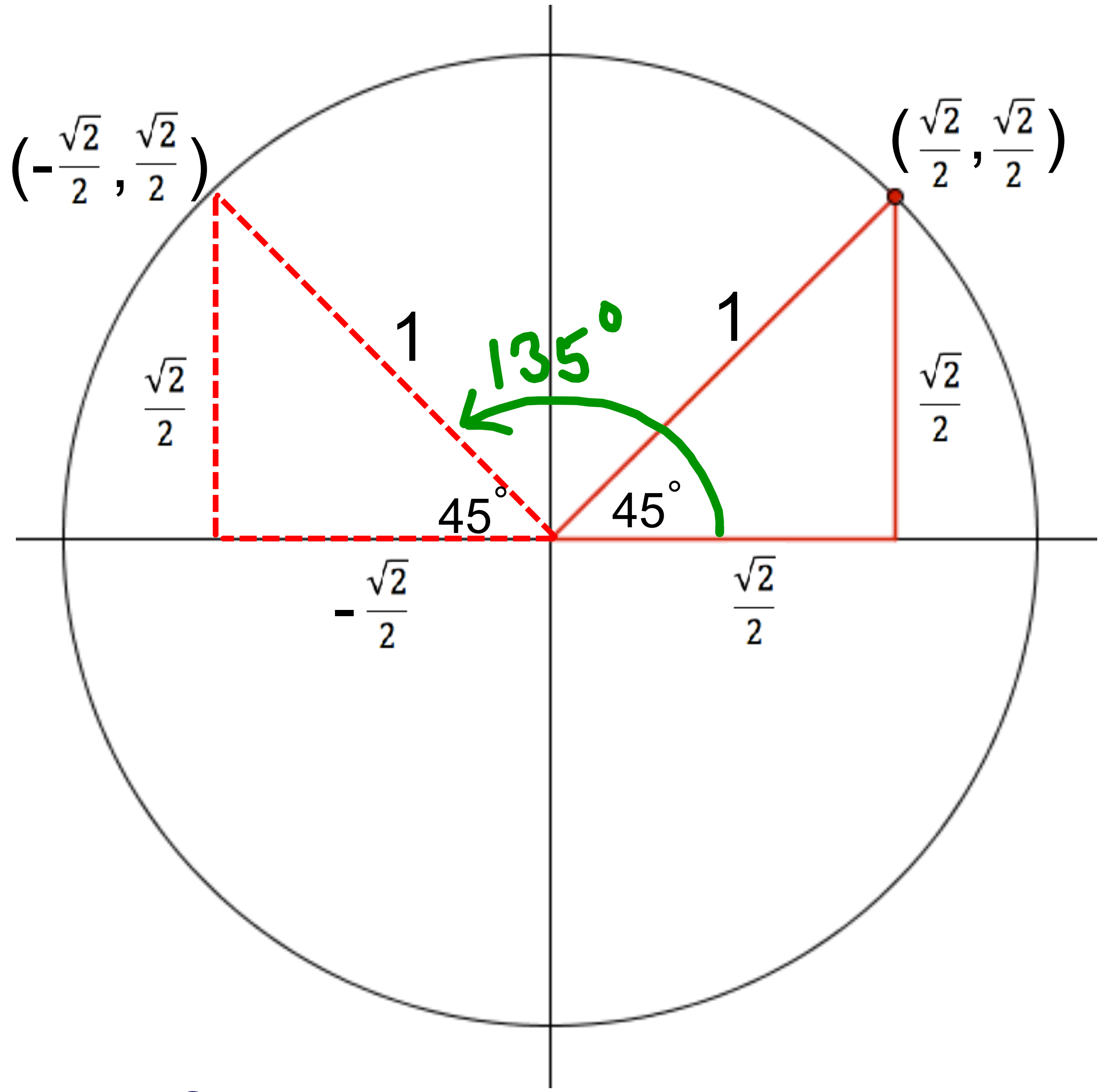
$$\cos(45^\circ) = \frac{\sqrt{2}}{2}$$

$$\sin(45^\circ) = \frac{\sqrt{2}}{2}$$

$$\tan(45^\circ) = 1$$

(see Triangle Trig Review unit for more detail on this topic)

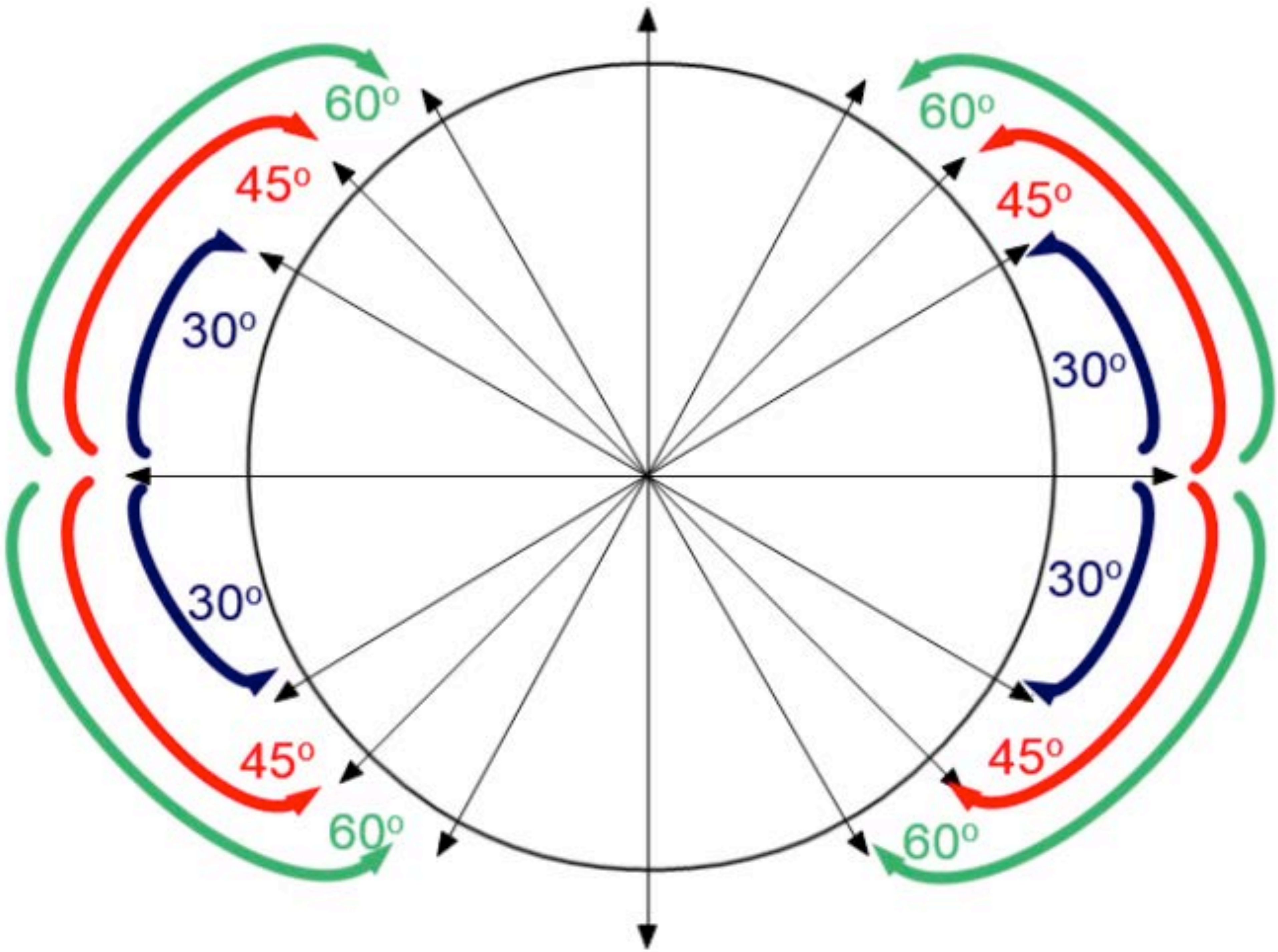
Special Triangles and the Unit Circle



Multiples of 45° angles have sin and cos of $\pm\frac{\sqrt{2}}{2}$,

depending on the quadrant.

Unit Circle



Unit Circle

Drag the degree and radian angle measures to the angles of the circle:

0°

45°

90°

135°

180°

225°

270°

315°

360°

0

π

$\frac{\pi}{4}$

2π

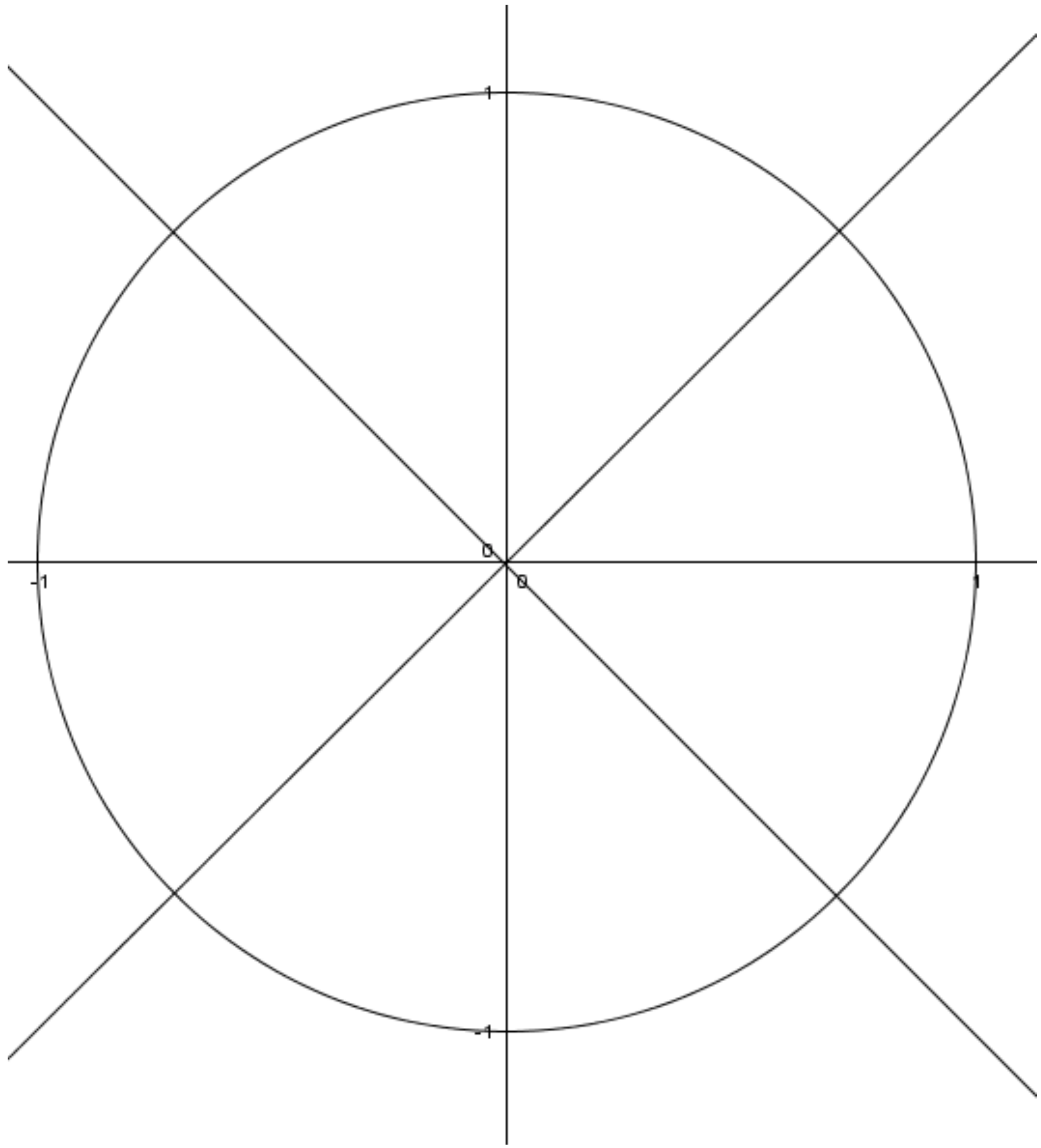
$\frac{5\pi}{4}$

$\frac{\pi}{2}$

$\frac{3\pi}{2}$

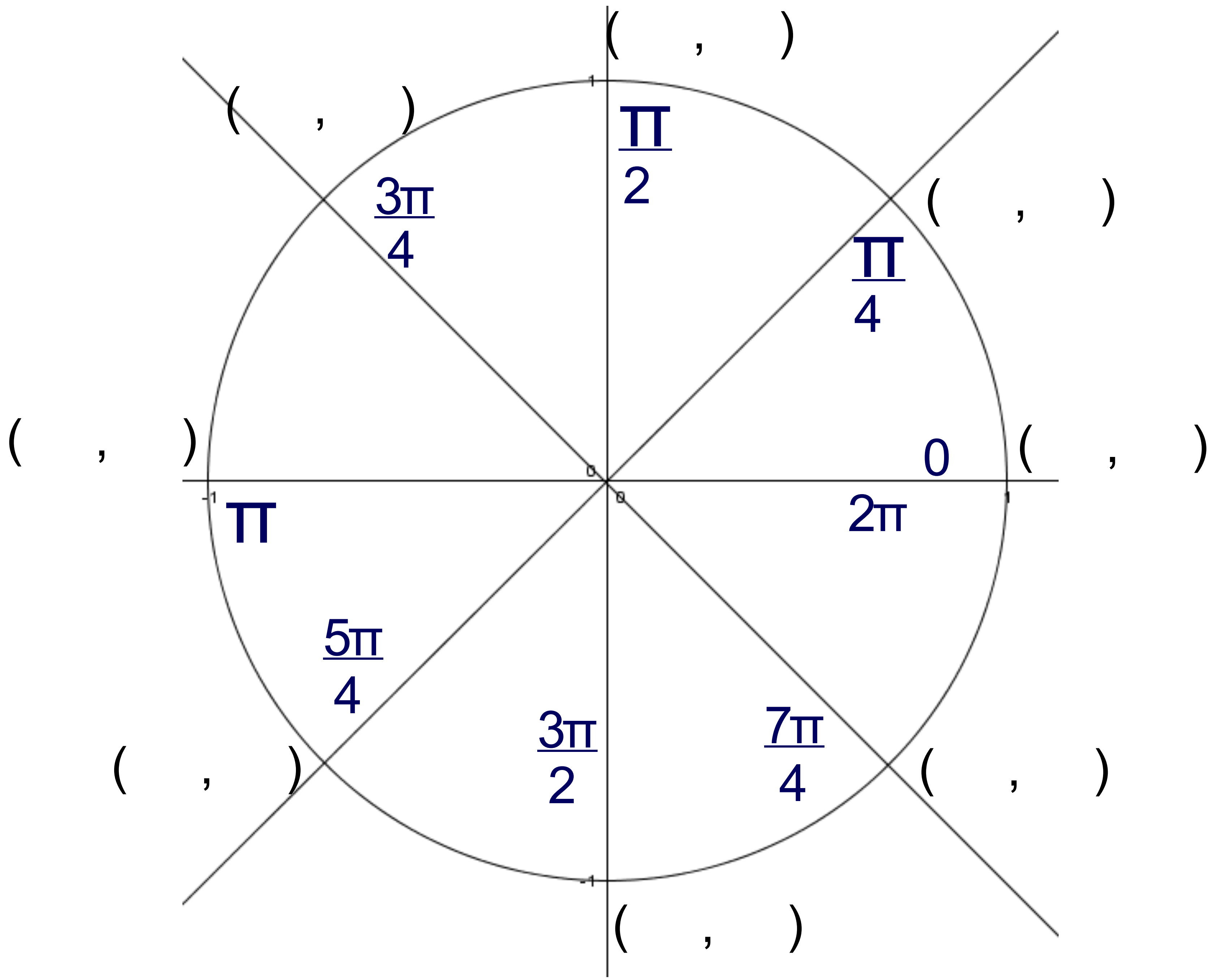
$\frac{3\pi}{4}$

$\frac{7\pi}{4}$



Unit Circle

Fill in the coordinates of x and y for each point on the unit circle:



$-\frac{\sqrt{2}}{2}$

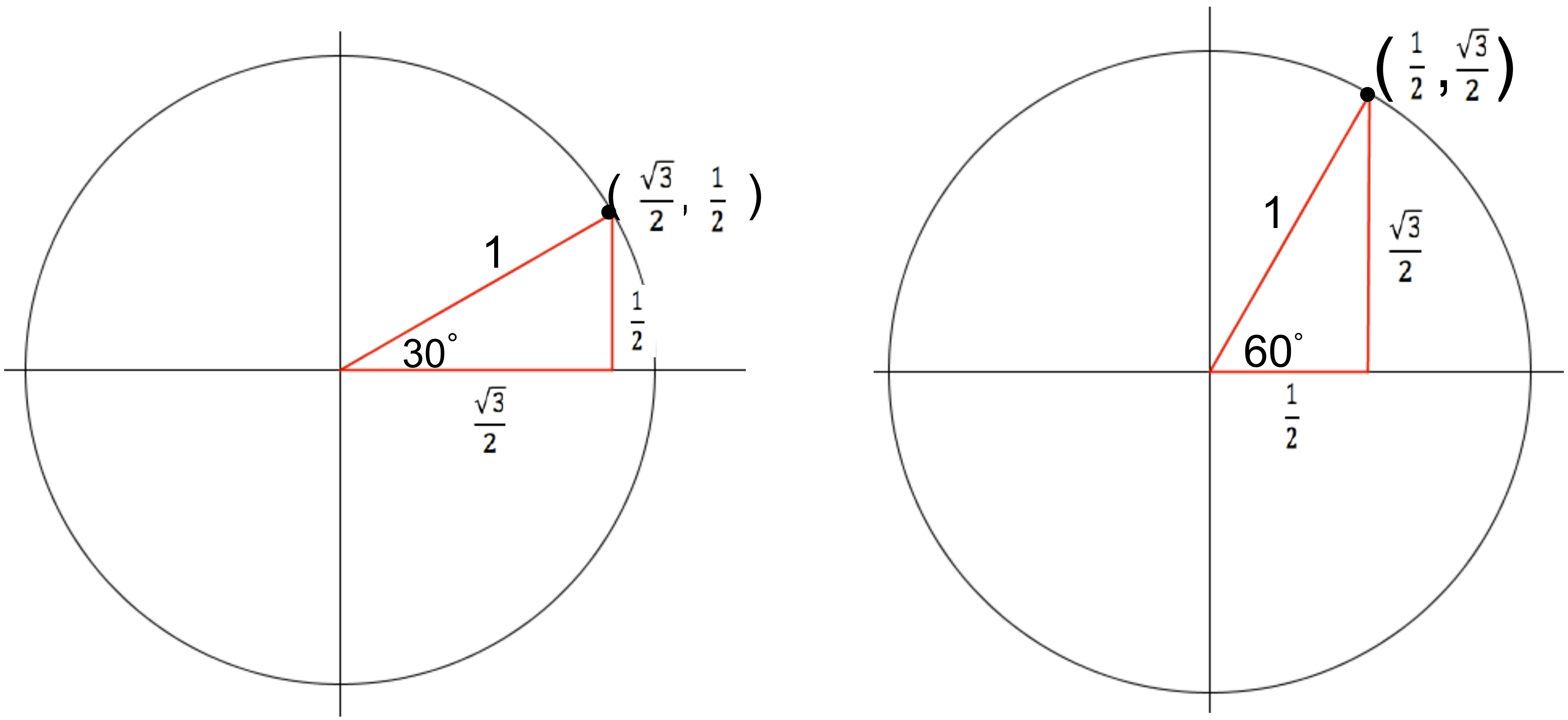
$\frac{\sqrt{2}}{2}$

0

1

-1

Special Triangles and the Unit Circle



Angles that are multiples of 30° have sin and cos

of $\pm \frac{1}{2}$ and $\pm \frac{\sqrt{3}}{2}$

Unit Circle

Drag the degree and radian angle measures to the angles of the circle:

0° $\frac{5\pi}{6}$ $\frac{7\pi}{6}$

30°

60° $\frac{\pi}{2}$ $\frac{11\pi}{6}$

90°

120° π $\frac{5\pi}{3}$

150° $\frac{\pi}{6}$

180° 0

210° 2π

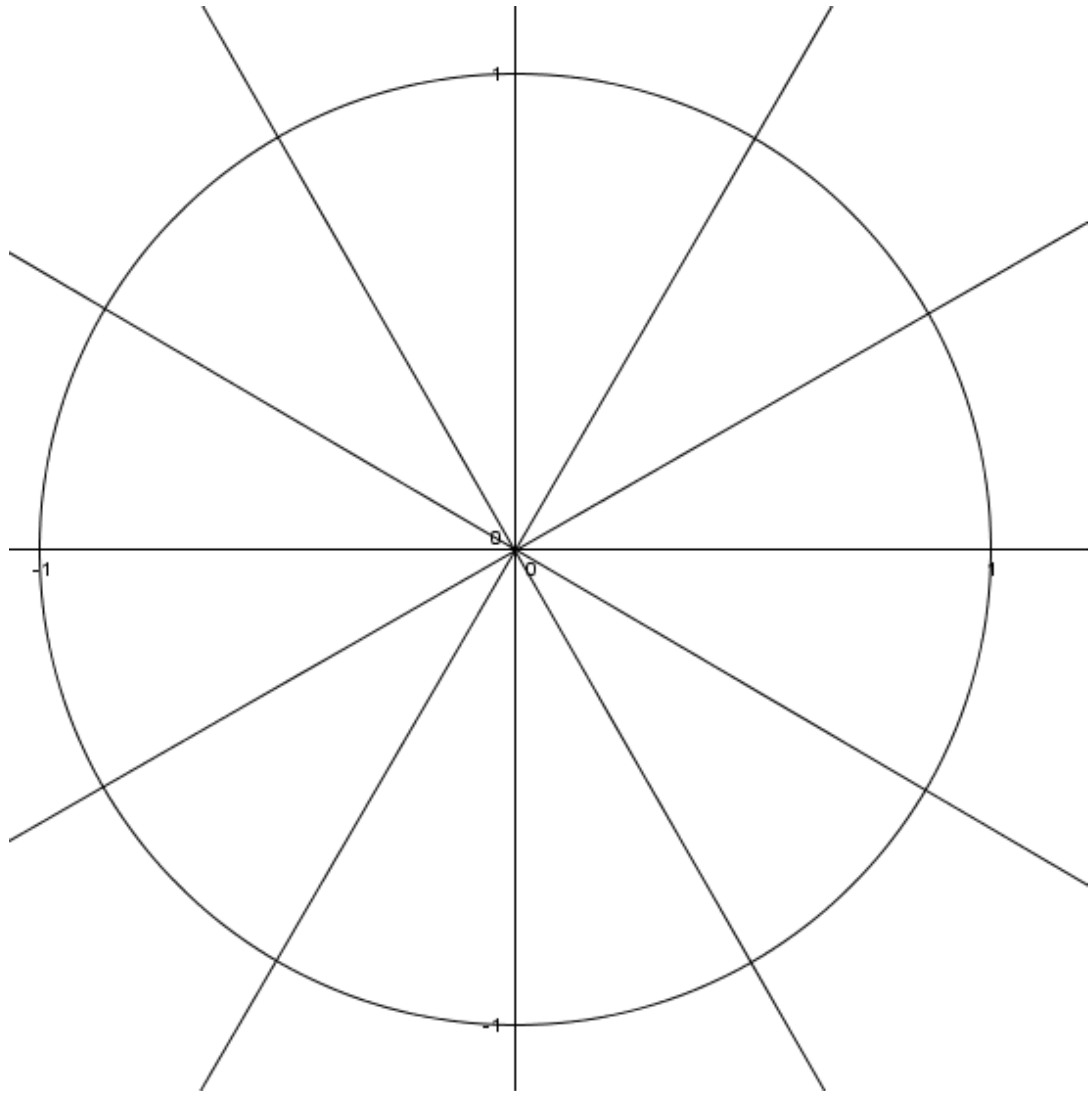
240° $\frac{4\pi}{3}$

270°

300° $\frac{\pi}{3}$

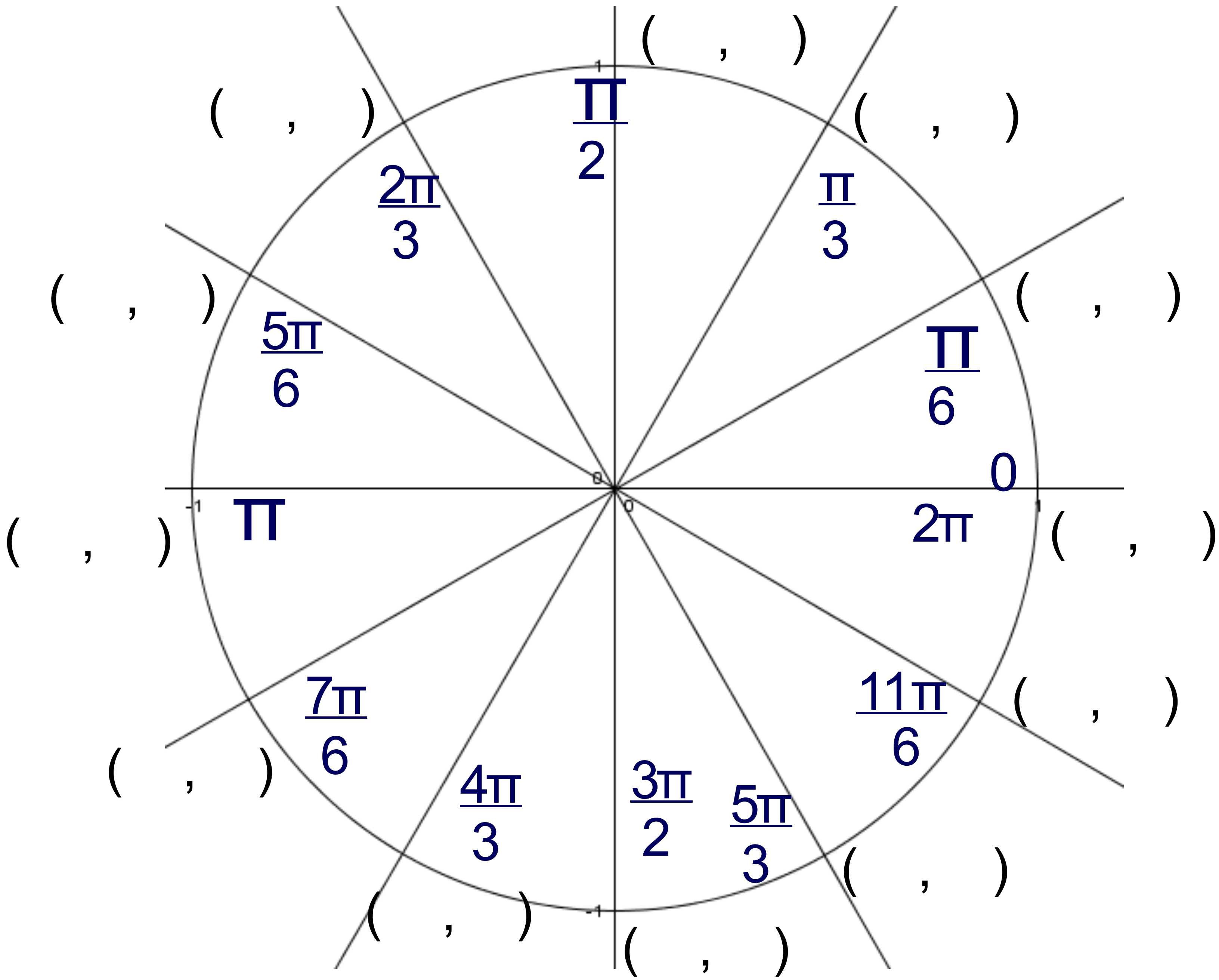
330° $\frac{3\pi}{2}$

360° $\frac{2\pi}{3}$



Unit Circle

Drag in the coordinates of x and y for each point on the unit circle:



$\frac{1}{2}$

$-\frac{1}{2}$

$\frac{\sqrt{3}}{2}$

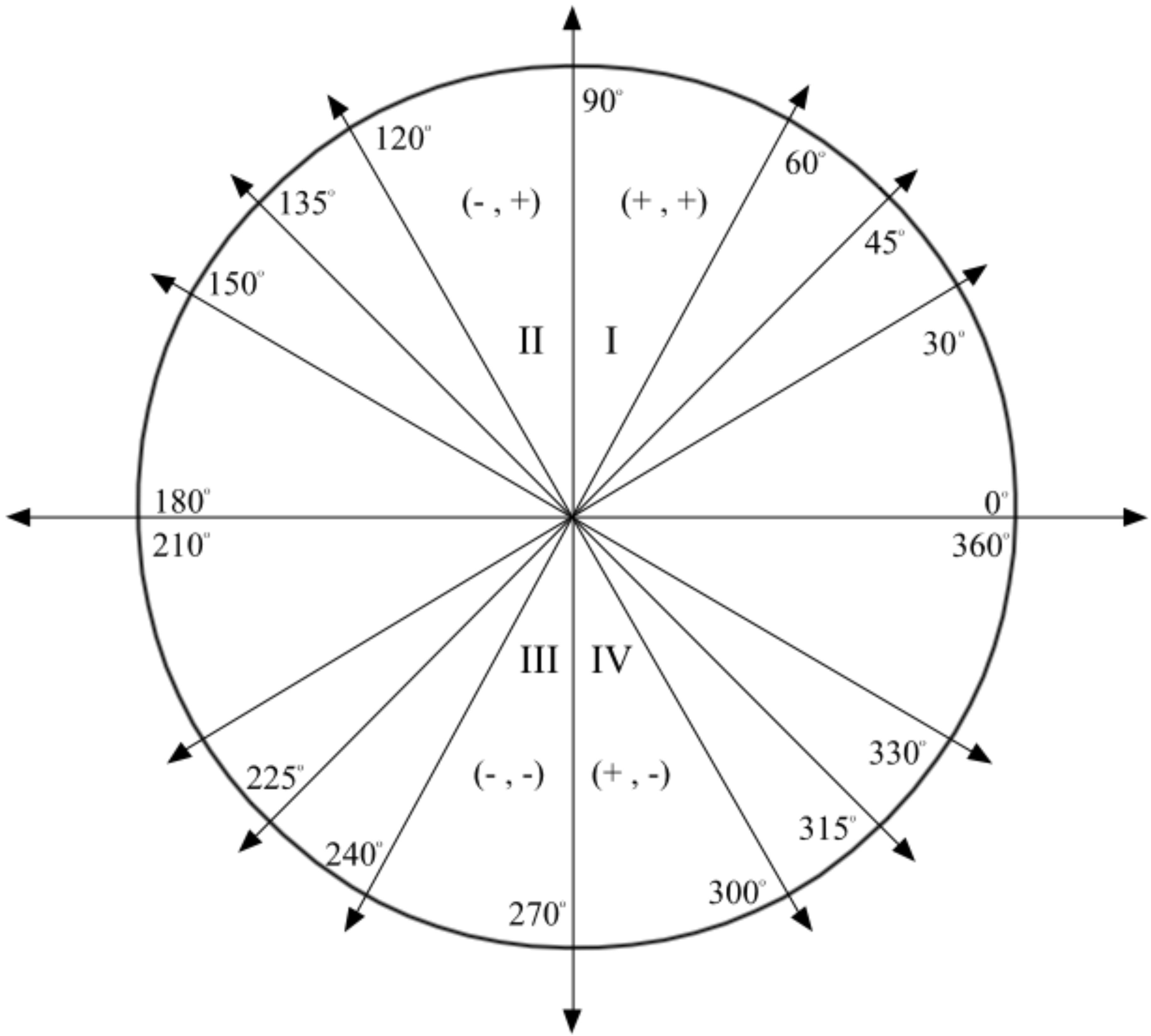
$-\frac{\sqrt{3}}{2}$

0

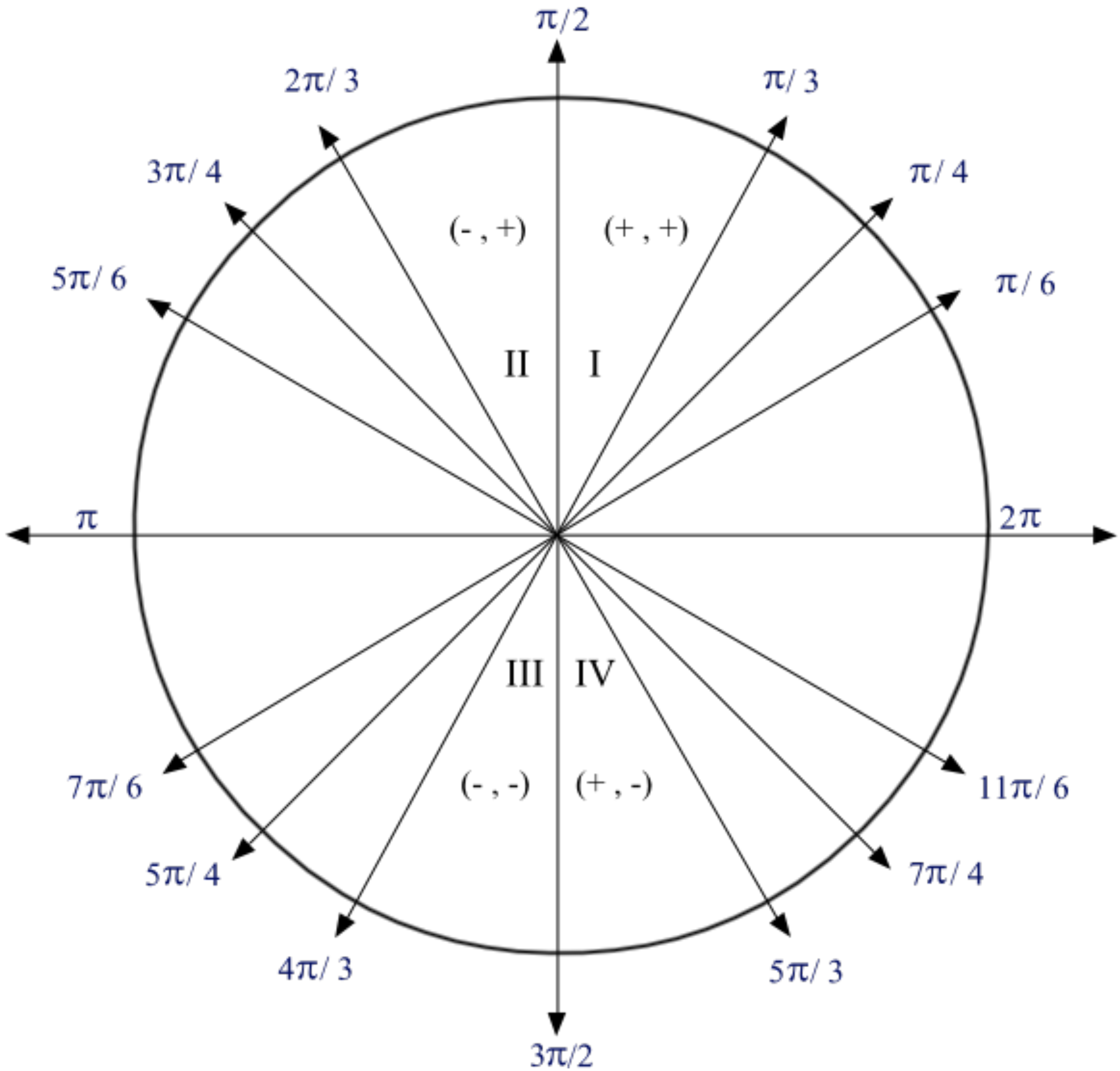
1

-1

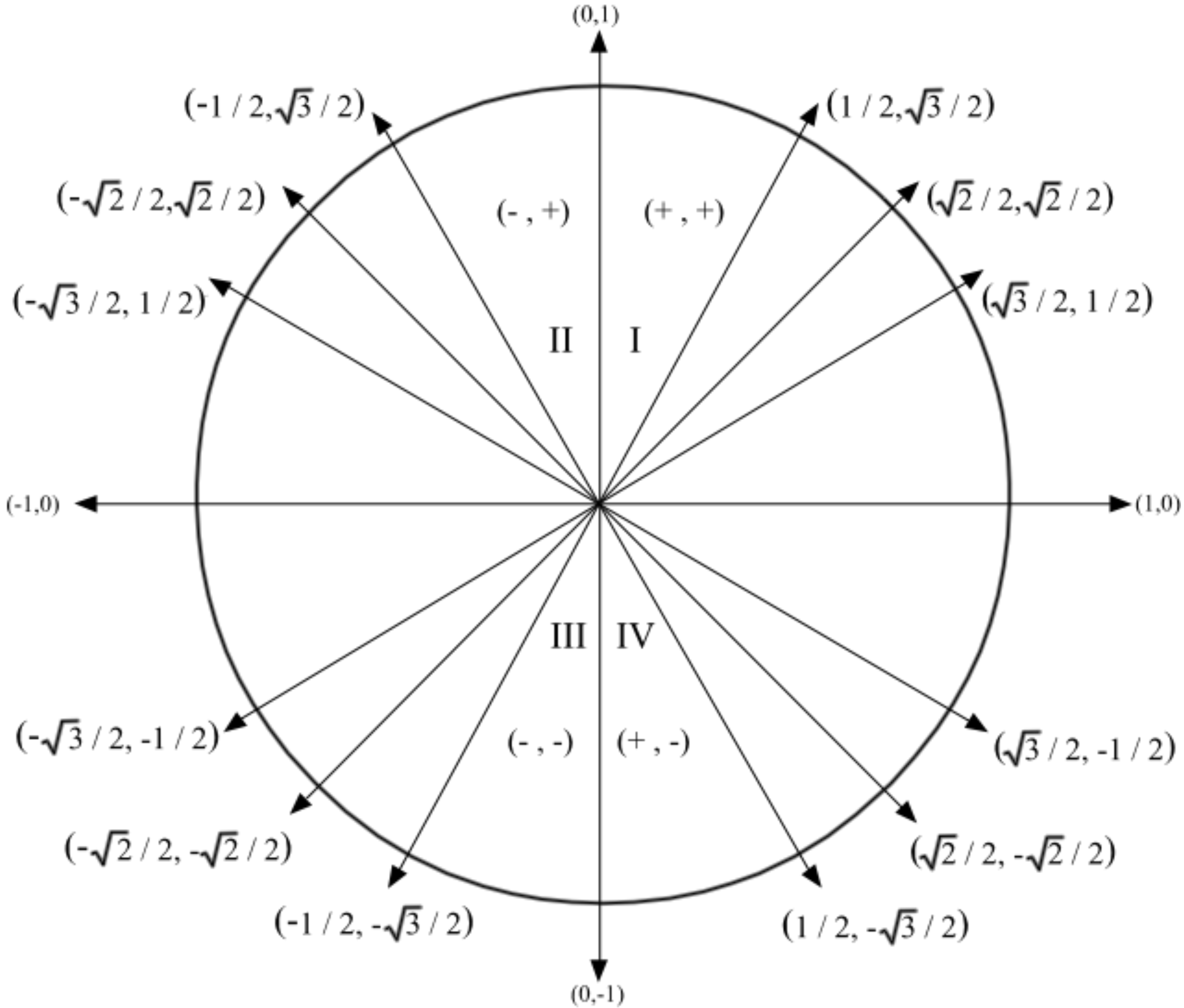
Special Angles in Degrees



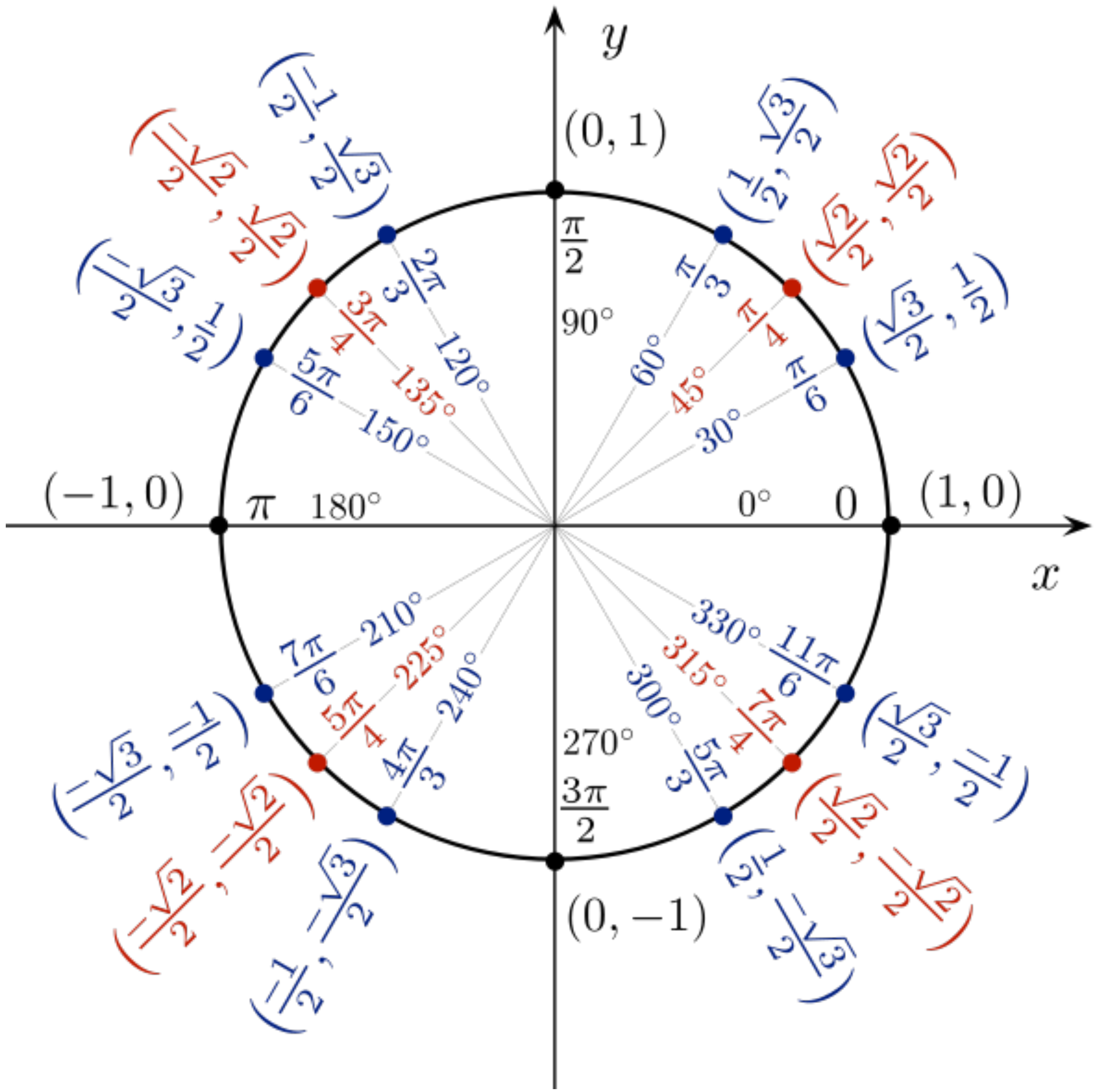
Radian Values of Special Angles



Exact Values of Special Angles



Put it all together...



Exact Values of Special Angles

Complete the table below:

Degrees	0°	30°	45°	60°	90°
Radians					
$\sin\ \theta$					
$\cos\ \theta$					
$\tan\ \theta$					

Exact Values of Special Angles

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
$0° = 0$	1	0	0
$30° = \frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60° = \frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$90° = \frac{\pi}{2}$	0	1	*undefined*
$120° = \frac{2\pi}{3}$	$\frac{-1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$135° = \frac{3\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$150° = \frac{5\pi}{6}$	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{-\sqrt{3}}{3}$
$180° = \pi$	-1	0	0

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
$210° = \frac{7\pi}{6}$	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	$\frac{\sqrt{3}}{3}$
$225° = \frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	1
$240° = \frac{4\pi}{3}$	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	$\sqrt{3}$
$270° = \frac{3\pi}{2}$	0	-1	*undefined*
$300° = \frac{5\pi}{3}$	$\frac{1}{2}$	-1	$-\sqrt{3}$
$315° = \frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	-1
$330° = \frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{3}$
$360° = 2\pi$	1	0	0

Exact Values of Special Angles

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

θ	$\sec\theta$	$\csc\theta$	$\cot\theta$
$0^\circ = 0$	1	*undefined*	*undefined*
$30^\circ = \frac{\pi}{6}$	$\frac{2\sqrt{3}}{3}$	2	$\sqrt{3}$
$45^\circ = \frac{\pi}{4}$	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ = \frac{\pi}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$
$90^\circ = \frac{\pi}{2}$	*undefined*	1	0
$120^\circ = \frac{2\pi}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$\frac{-\sqrt{3}}{3}$
$135^\circ = \frac{3\pi}{4}$	$-\sqrt{2}$	$\sqrt{2}$	-1
$150^\circ = \frac{5\pi}{6}$	$-\frac{2\sqrt{3}}{3}$	2	$-\sqrt{3}$
$180^\circ = \pi$	-1	*undefined*	*undefined*

θ	$\sec\theta$	$\csc\theta$	$\cot\theta$
$210^\circ = \frac{7\pi}{6}$	$-\frac{2\sqrt{3}}{3}$	-2	$\sqrt{3}$
$225^\circ = \frac{5\pi}{4}$	$-\sqrt{2}$	$-\sqrt{2}$	1
$240^\circ = \frac{4\pi}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$
$270^\circ = \frac{3\pi}{2}$	*undefined*	-1	0
$300^\circ = \frac{5\pi}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$\frac{-\sqrt{3}}{3}$
$315^\circ = \frac{7\pi}{4}$	$\sqrt{2}$	$-\sqrt{2}$	-1
$330^\circ = \frac{11\pi}{6}$	$\frac{2\sqrt{3}}{3}$	-2	$-\sqrt{3}$
$360^\circ = 2\pi$	1	*undefined*	*undefined*

Finding Angles and Trig Values

If we know one trig function value and the quadrant in which the angle lies, we can find the angle and the other trig values.

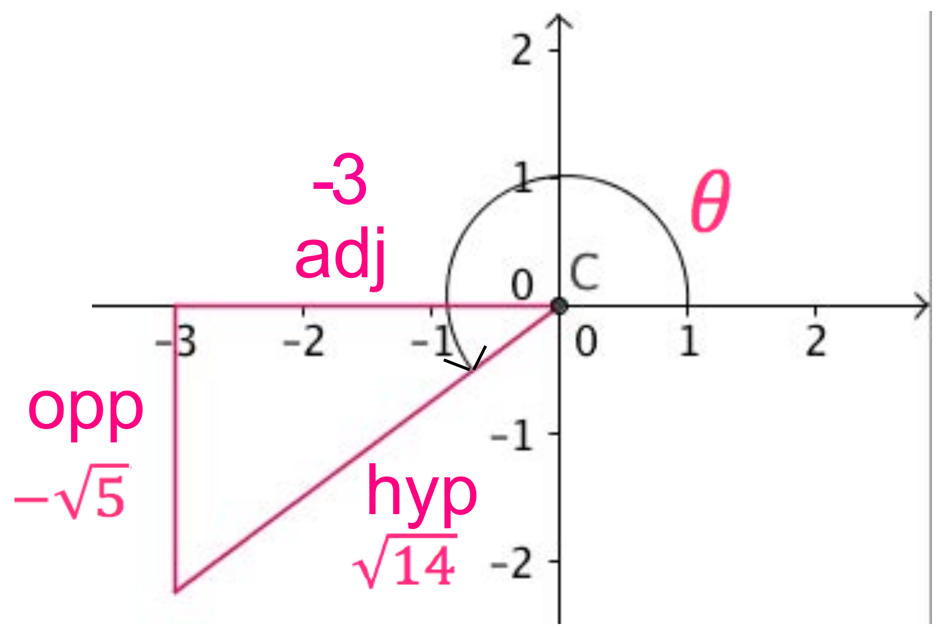
Finding Angles and Trig Values

Example: If $\tan\theta = \frac{\sqrt{5}}{3}$, and $\sin\theta < 0$, find $\sin\theta$, $\cos\theta$ and the value of θ.

Solution: Since $\tan\theta$ is positive and $\sin\theta$ is negative, the terminal side of θ must be in Quadrant III.

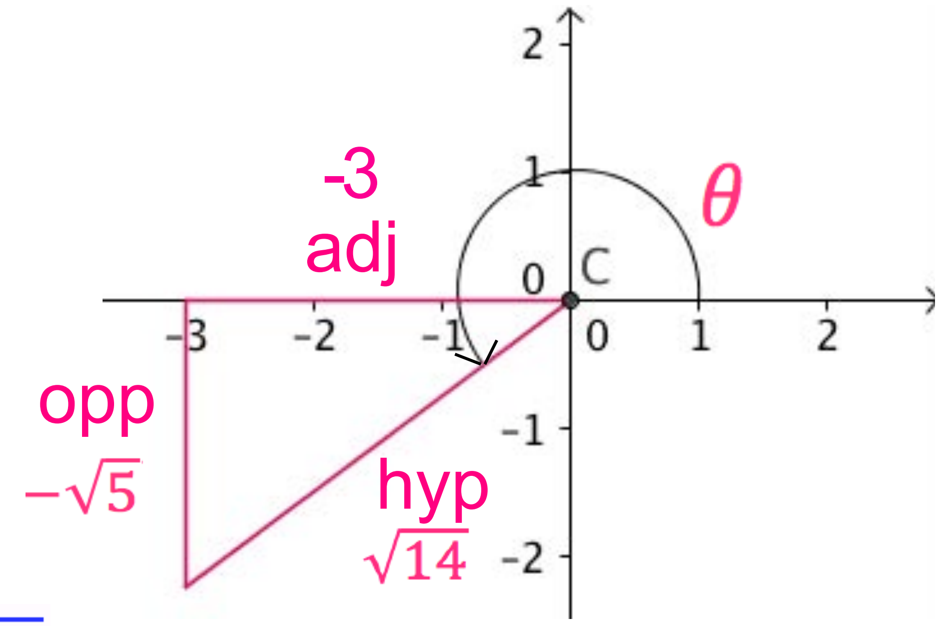
- Draw a right triangle in
- Quadrant III.
- Use the Pythagorean
- Theorem to find the
- length of the
- hypotenuse:

$$(-\sqrt{5})^2 + (-3)^2 = hyp^2$$
$$5 + 9 = hyp^2$$
$$14 = hyp^2$$
$$hyp = \sqrt{14}$$



(Continued on next slide)

Finding Angles and Trig Values



Once we know the lengths for each side, we can calculate the sin, cos and the angle. Used the signed numbers to get the correct values.

$$\sin \theta = \frac{-\sqrt{5}}{\sqrt{14}} \quad \frac{-\sqrt{70}}{14}$$

$$\cos \theta = \frac{-3}{\sqrt{14}} = -\frac{3\sqrt{14}}{14}$$

Use any inverse trig function to find the angle. tan-1($\frac{\sqrt{5}}{3}$) ≈ 36.7°. Because the angle is in QIII, we need to add 180 + 36.7 = 216.7, °

so θ ≈ 217°.

19 $\cos\frac{2\pi}{3}$

A 1

B $\frac{\sqrt{3}}{2}$

C $\frac{\sqrt{2}}{2}$

D $\frac{1}{2}$

E 3

F -1

G $-\frac{\sqrt{3}}{2}$

H $-\frac{\sqrt{2}}{2}$

I $-\frac{1}{2}$

J 0

Answer

20 $\sin\frac{5\pi}{3}$

A 1

B $\frac{\sqrt{3}}{2}$

C $\frac{\sqrt{2}}{2}$

D $\frac{1}{2}$

E 3

F -1

G $-\frac{\sqrt{3}}{2}$

H $-\frac{\sqrt{2}}{2}$

I $-\frac{1}{2}$

J 0

Answer

21 $\tan\frac{4\pi}{3}$

A 1

B $\frac{\sqrt{3}}{3}$

C $\frac{\sqrt{2}}{2}$

D *undefined*

E $\sqrt{3}$

F -1

G $-\frac{\sqrt{3}}{3}$

H $-\frac{\sqrt{2}}{2}$

I $-\sqrt{3}$

J 0

Answer

22 Which functions are positive in the second quadrant? Choose all that apply.

A cos x

B sin x

C tan x

D sec x

E csc x

F cot x

Answer

23 Which functions are positive in the fourth quadrant? Choose all that apply.

A cos x

B sin x

C tan x

D sec x

E csc x

F cot x

Answer

24 Which functions are positive in the third quadrant? Choose all that apply.

A cos x

B sin x

C tan x

D sec x

E csc x

F cot x

Answer